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# Three-point functions in $\mathcal{N} = 4$ SYM theory at one-loop

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Abstract: We analyze the one-loop correction to the three-point function coefficient of scalar primary operators in $\mathcal{N} = 4$ SYM theory. By applying constraints from the superconformal symmetry, we demonstrate that the type of Feynman diagrams that contribute depends on the choice of renormalization scheme. In the planar limit, explicit expressions for the correction are interpreted in terms of the hamiltonians of the associated integrable closed and open spin chains. This suggests that at least at one-loop, the planar conformal field theory is integrable with the anomalous dimensions and OPE coefficients both obtainable from integrable spin chain calculations. We also connect the planar results with similar structures found in closed string field theory.

Keywords: AdS-CFT and dS-CFT Correspondence, Conformal and W Symmetry, Extended Supersymmetry, Integrable Field Theories.



## Contents





## 1. Introduction

Integrability in quantum field theory has for the most part been relegated to the realms of two dimensional theories. (For an overview, see for example, [1]). Recently, there has been much excitement over the prospects of integrability in four dimensional gauge theory. Much of the recent work has focused on $\mathcal{N} = 4$ super Yang-Mills (SYM) theory. This theory is special in that it is superconformal and is widely believed to be dual to type-IIB string theory on $AdS_5 \times S^5$ by the AdS/CFT correspondence.

In the large-$N$ planar limit of $\mathcal{N} = 4$ SYM, an important observation was put forth by Minahan and Zarembo [2] concerning the first order $\lambda = g^2_{\rm YM}N$ correction to the scaling dimension of composite single trace operators consisting of derivative-free scalar fields. They pointed out that these operators can be naturally mapped into states of an integrable

SO(6) spin chain. And amazingly, the hamiltonian of the spin chain is proportional to the $O(\lambda)$ correction to the scaling dimension of the operators. In follow up works, Beisert, et al. [3, 4, 5] extended this relation to arbitrary single trace composite operators and showed that the first order correction to the dilation operator is in fact given by the hamiltonian of the integrable PSU$(2,2|4)$ spin chain.

Further hints of $\mathcal{N}=4$ SYM planar integrability has also emerged via the AdS/CFT correspondence. Type IIB theory on $AdS_5 \times S^5$ has been found to contain an infinite number of nonlocal charges [6, 7, 8].[1] And in the extensive analysis of spinning strings solutions, integrable structures have played a central role (see [10] for a review). Yet, with all these suggestions, the origin of this apparent planar integrability in $\mathcal{N}=4$ SYM theory is still not clear.

So far, evidences of planar integrability have all been gathered from analyses of the scaling dimension spectrum of operators in the theory. But if the planar $\mathcal{N}=4$ SYM theory is indeed integrable, then the dynamical aspects of the theory must also be describable by integrable structures. For a conformal field theory, the dynamical data are encoded in the coefficients of the operator product expansions (OPE), i.e. the structure constants. These structure constants, directly related to the coefficient of the three-point functions, are important for solving the correlation functions of the theory.

In this paper, we analyze the one-loop correction to the three-point function coefficients of conformal primary operators. We will neglect possible additional $O(\lambda)$ correction from $\lambda$ dependent operator mixing that can only be seen at the two-loop level [11].[2] As with Minahan and Zarembo, we will consider only scalar conformal primaries in the SO(6) subsector of the theory and emphasize the planar limit.[3] Our calculations are much simplified by exploiting constraints from the superconformal symmetry of the theory. As is known from the study of non-renormalization of two- and three-point functions of BPS operators, $\mathcal{N}=4$ supersymmetry relates functional dependence of various $O(\lambda)$ Feynman diagrams [15]. Imposing further the constraints from conformal invariance, the first order correction to the three-point coefficients can be obtained, surprisingly, without summing all possible Feynman diagrams. As we will show, the first order correction may be obtained just by summing up only two-point Feynman diagram interactions (i.e. diagrams that connect two operators).[4] Alternatively, one can choose to sum up mainly three-point Feynman diagram interactions. These two prescriptions are just the result of working in two special renormalization schemes. Since the physical correction must be scheme independent, the two prescriptions provide us with two different, but complementary, pictures of the three-point function interaction.

In the planar limit, the two-point Feynman diagram description leads naturally to a spin chain interpretation. For the generic non-extremal three-point function, the one-loop correction to the three-point coefficient turns out to involve the hamiltonian of both closed

[1] A discussion of nonlocal charges in free $\mathcal{N}=4$ SYM theory can be found in [9].

[2] A discussion of these additional contributions is given in section 7.

[3] Some previous calculations of non-protected $\mathcal{N}=4$ SYM scalar three-point functions can be found in [12, 13, 14].

[4] This has been utilized in calculations of three-point functions of BPS operators in [16].

and open SO(6) integrable spin chains. Open spin chains can arise by splitting a periodic closed spin chain into two "correlated" open spin chains. As we will show, the three primary operators can be combined to give three open spin chain density matrices. Indeed, the one-loop correction contains a contribution from the ensemble average of the open spin chain hamiltonian with respect to the three density matrices. As for the three-point Feynman diagram description, it does not seem to have a direct interpretation using conventional spin chain language. Its natural setting seems to be that of string field theory. We shall identify the planar three-point Feynman diagram prescription with an analogous structure, the Witten type three-string vertex, in covariant closed string field theory.

In section 2, we provide the general form of two- and three-point functions of conformal primary operators at one-loop. In section 3, we first explain how supersymmetry relates various Feynman diagrams. Although our presentation will mainly focus on the planar limit, we will utilize and point out results that are valid at finite $N$. We proceed to apply the conformal symmetry constraints to calculate the three-point function coefficient. Of interest is that the one-loop correction of the extremal three-point function coefficient depends only on the anomalous dimensions. In section 4, we provide explicit results in the planar limit. The planar results are interpreted in the context of integrable spin chains in section 5. In section 6, from our SYM results, we infer some properties of closed string field theory in $AdS_5 \times S^5$. We close with a discussion in section 7. Appendix A contains the computation of integrals that appear in the contributing Feynman diagrams. And in appendix B, we give a matrix integral representation of SO(6) index structure of three-point functions.

## 2. General form of two-point and three-point function at one-loop

Let $\mathcal{O}^B$ denote bare scalar conformal primary operators. For bare operators with identical free scaling dimension $\Delta_0$, the two-point function to first order in $\lambda = g_{\rm YM}^2 N$ takes the form,

$$\begin{aligned}\langle \bar{\mathcal{O}}^B_\alpha(x_1)\mathcal{O}^B_\beta(x_2)\rangle &= \frac{1}{|x_{12}|^{2\Delta_0}}\left[g_{\alpha\beta} - h_{\alpha\beta}\,\ln|x_{12}\Lambda|^2\right] \\ &= \frac{1}{|x_{12}|^{2\Delta_0}} g_{\alpha\rho}\left[\delta^\rho_\beta - g^{\rho\sigma}h_{\sigma\beta}\,\ln|x_{12}\Lambda|^2\right]\end{aligned} \tag{2.1}$$

where $x_{12}^\mu = x_1^\mu - x_2^\mu$, $(g^{-1}h)^\alpha{}_\beta$ is the anomalous dimension matrix, and $\Delta_{0\alpha} = \Delta_{0\beta} = \Delta_0$. Note that both $g_{\alpha\beta}$ and $h_{\alpha\beta}$ may contain terms of $O(\lambda)$. By a linear transformation, $\mathcal{O}^B_\alpha \to \mathcal{O}_\alpha = \mathcal{O}^B_\rho\, U^\rho{}_\alpha$, the matrices can be diagonalized as follows [17, 18],

$$\begin{aligned}(U^\dagger)_\alpha{}^\rho\, g_{\rho\sigma}\, U^\sigma{}_\beta &= \delta_{\alpha\beta}\, A_\alpha \\ (U^{-1})^\alpha{}_\rho\,(g^{-1}h)^\rho{}_\sigma\, U^\sigma{}_\beta &= \delta^\alpha{}_\beta\,\gamma_\alpha\,.\end{aligned} \tag{2.2}$$

Here, $A_\alpha$ and $\gamma_\alpha$ are the normalizations and anomalous dimensions, respectively. Notice that at the one-loop level, $U^\rho{}_\alpha$ is independent of $\lambda$; therefore, $\lambda$ dependent operator mixing does not appear at one-loop. We will decompose the normalization as $A_\alpha = N_\alpha^2\,[1 +$

$2a_\alpha\,\lambda + O(\lambda^2)]$, where $a_\alpha$ is a scheme dependent constant. The two-point function for an orthogonalized operator (or eigen-operator) $\mathcal{O}_\alpha$ then becomes

$$\begin{aligned}\langle \bar{\mathcal{O}}_\alpha(x_1)\mathcal{O}_\alpha(x_2)\rangle &= \frac{N_\alpha^2[1+2a_\alpha\,\lambda]}{|x_{12}|^{2\Delta_0}}\left[1-\gamma_\alpha\,\ln|x_{12}\Lambda|^2\right]\\ &= \frac{N_\alpha^2}{|x_{12}|^{2\Delta_0}}\left[1+2a_\alpha\,\lambda-\gamma_\alpha\,\ln|x_{12}\Lambda|^2\right]. \end{aligned} \tag{2.3}$$

From the bare eigen-operator, the renormalized operator is defined to be

$$\tilde{\mathcal{O}}_\alpha = \mathcal{O}_\alpha\left[1-a_\alpha\,\lambda+\gamma_\alpha\ln\left|\frac{\Lambda}{\mu}\right|+O(\lambda^2)\right] \tag{2.4}$$

where $\mu$ is the renormalization scale. As required by conformal invariance, the two-point function of renormalized primary operators takes the form,

$$\langle \bar{\tilde{\mathcal{O}}}_\alpha(x_1)\tilde{\mathcal{O}}_\alpha(x_2)\rangle = \frac{N_\alpha^2}{|x_{12}|^{2\Delta_{0\alpha}}|x_{12}\,\mu|^{2\gamma_\alpha}} \tag{2.5}$$

with scaling dimension $\Delta_\alpha = \Delta_{0\alpha}+\gamma_\alpha$. Note that $\tilde{\mathcal{O}}_\alpha$ is conventionally defined such that $N_\alpha = 1$. For convenience, we will only require that the renormalized operator be orthogonalized and not orthonormalized.

Conformal invariance constrains the three-point function for renormalized primary operators to be

$$\langle \bar{\tilde{\mathcal{O}}}_\alpha(x_1)\tilde{\mathcal{O}}_\beta(x_2)\tilde{\mathcal{O}}_\rho(x_3)\rangle = \frac{N_\alpha N_\beta N_\rho\, c_{\alpha\beta\rho}}{|x_{12}|^{\Delta_\alpha+\Delta_\beta-\Delta_\rho}|x_{13}|^{\Delta_\alpha+\Delta_\rho-\Delta_\beta}|x_{23}|^{\Delta_\beta+\Delta_\rho-\Delta_\alpha}\,|\mu|^{\gamma_\alpha+\gamma_\beta+\gamma_\rho}} \tag{2.6}$$

where $c_{\alpha\beta\rho}$ is the three-point function coefficient. We are interested in finding the one-loop correction to the structure constant. Therefore, we decompose

$$c_{\alpha\beta\rho} = c^0_{\alpha\beta\rho}\Big(1+\lambda\, c^1_{\alpha\beta\rho}+O(\lambda^2)\Big) \tag{2.7}$$

assuming $c^0_{\alpha\beta\rho}\neq 0$. Substituting (2.4) into (2.6) , the three-point function for bare eigen-operators is given at one-loop by

$$\begin{aligned}\langle \bar{\mathcal{O}}_\alpha(x_1)\mathcal{O}_\beta(x_2)\mathcal{O}_\rho(x_3)\rangle &= \frac{N_\alpha N_\beta N_\rho\, c^0_{\alpha\beta\rho}(1+\lambda\, c^1_{\alpha\beta\rho})\,[1+\lambda(a_\alpha+a_\beta+a_\rho)]}{|x_{12}|^{\Delta_\alpha+\Delta_\beta-\Delta_\rho}|x_{13}|^{\Delta_\alpha+\Delta_\rho-\Delta_\beta}|x_{23}|^{\Delta_\beta+\Delta_\rho-\Delta_\alpha}\,|\Lambda|^{\gamma_\alpha+\gamma_\beta+\gamma_\rho}}\\ &= \frac{C^0}{|x_{12}|^{\Delta_{0\alpha}+\Delta_{0\beta}-\Delta_{0\rho}}|x_{13}|^{\Delta_{0\alpha}+\Delta_{0\rho}-\Delta_{0\beta}}|x_{23}|^{\Delta_{0\beta}+\Delta_{0\rho}-\Delta_{0\alpha}}}\times\\ &\quad\times\left(1+\lambda C^1-\gamma_\alpha\ln\left|\frac{x_{12}x_{13}\Lambda}{x_{23}}\right|-\gamma_\beta\ln\left|\frac{x_{12}x_{23}\Lambda}{x_{13}}\right|-\gamma_\rho\ln\left|\frac{x_{23}x_{13}\Lambda}{x_{12}}\right|\right)\end{aligned} \tag{2.8}$$

where we have defined $C^0 = N_\alpha N_\beta N_\rho c^0_{\alpha\beta\rho}$ as the overall factor and $C^1 = c^1_{\alpha\beta\rho}+a_\alpha+a_\beta+a_\rho$ as the constant one-loop correction. Notice that $C^1$ is dependent on the renormalization scheme. However, we are interested in calculating $c^1_{\alpha\beta\rho} = C^1-a_\alpha-a_\beta-a_\rho$ which is scheme independent.

If desired, we can transform the three-point expression in (2.8) into that for bare operators, $\mathcal{O}^B$, by applying the linear transformation, $\mathcal{O}_\alpha = \mathcal{O}^B_\rho\, U^\rho{}_\alpha$.

## 3. Perturbative calculations of three-point functions

We write the euclidean $\mathcal{N}=4$ action with SU($N$) gauge symmetry as

$$\mathcal{S}=\frac{N}{\lambda}\int d^4x\,\mathrm{tr}\left\{\frac{1}{2}F_{\mu\nu}F^{\mu\nu}+D^\mu\phi^i D_\mu\phi^i-\frac{1}{2}[\phi^i,\phi^j]^2+\text{fermions terms}\right\} \tag{3.1}$$

with the scalar fields $\phi^i$, $i=1,\ldots,6$, in the **6** of the SO(6) ($R$ symmetry group). We will only consider conformal primary operators composed of derivative-free scalar fields. At finite $N$, the operators of interest must belong to the closed SU(2) subsector of the theory [3, 4]. These are operators that can be expressed in terms of only two complex scalar fields, for example, $Z=\phi^1+i\phi^2$ and $W=\phi^3+i\phi^4$. By SO(6) representation theory arguments, such operators do not mix with scalar combinations of fermion bilinears, field strengths $F_{\mu\nu}$, or covariant derivatives $D_\mu$ [3]. On the other hand, when working in the planar limit, we will choose to focus on the single trace operators in the SO(6) subsector. In the planar limit, single trace operators do not mix with multi-trace operators. This allows us to neglect subtleties that may arise due to the regularization of multi-trace operators [19] in the SO(6) sector. We will write SO(6) single trace operators as

$$\mathcal{O}[\psi_I]=\frac{1}{\lambda^{L/2}}\,\psi_{i_1\ldots i_L}\,\mathrm{tr}\,\phi^{i_1}\cdots\phi^{i_L} \tag{3.2}$$

where $I=\{i_1,\ldots,i_L\}$ and the constant coefficients $\psi_{i_1\ldots i_L}$ are independent of $N$ and $\lambda$. The additional $\lambda^{-L/2}$ factor in (3.2) is inserted so that the planar two-point functions $\langle\bar{\mathcal{O}}[\psi_I]\mathcal{O}[\psi_J]\rangle\sim O(N^0)$ and the planar three-point functions $\langle\bar{\mathcal{O}}[\psi_I]\mathcal{O}[\psi_J]\mathcal{O}[\psi_K]\rangle\sim O(N^{-1})$.

Below, we will first review some important characteristics of the functional forms of Feynman diagrams at one-loop found in the studies of two- and three-point functions of BPS operators [15, 20, 16]. Since these relations are independent of $N$, we will discuss the Feynman diagrams below in the planar limit. The planar Feynman diagrams will also play an important role in section 4. We will then utilize conformal symmetry to obtain general formulas at finite $N$ for $c^1_{\alpha\beta\rho}$, the one-loop correction to the three-point function coefficient defined in (2.7).

### 3.1 Feynman diagrams in the planar limit

For three-point functions, two classes of Feynman diagrams contribute. Contributions that arise already in the two-point function are shown in figure 1. In the planar limit, these

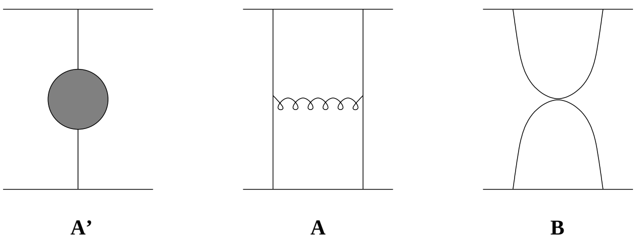
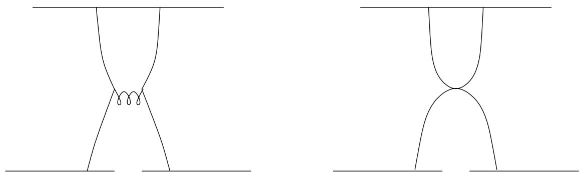



**Figure 1:** Two-point Feynman diagrams.



E F

**Figure 2:** Three-point Feynman diagrams.

interactions can at most be nearest-neighbor. Acting on two operators $\mathcal{O}[\psi_I](x_1)\mathcal{O}[\psi_J](x_2)$, the diagrams are expressed with emphasis on the SO(6) indices as follows.

$$\begin{aligned}
\text{Diagram A}' &= \delta^{j_l}_{i_l}\, A'(x_1,x_2)\, N^{-1}\, G(x_1,x_2) \\
\text{Diagram A} &= \delta^{j_l}_{i_l}\delta^{j_{l+1}}_{i_{l+1}}\, A(x_1,x_2)\, N^{-1}\, [G(x_1,x_2)]^2 \\
\text{Diagram B} &= \left(2\delta^{j_{l+1}}_{i_l}\delta^{j_l}_{i_{l+1}} - \delta^{j_l}_{i_l}\delta^{j_{l+1}}_{i_{l+1}} - \delta_{i_l i_{l+1}}\delta^{j_l j_{l+1}}\right) B(x_1,x_2)\, N^{-1}\, [G(x_1,x_2)]^2 \qquad (3.3)
\end{aligned}$$

where we have extracted factors of $1/N$ and the free scalar propagator $G(x_1,x_2)=\frac{\lambda}{N}\frac{1}{8\pi^2|x_{12}|^2}$ in defining the functions, $A'(x_1,x_2)$, $A(x_1,x_2)$, and $B(x_1,x_2)$.[5] Exact forms of these functions can be calculated directly. However, a relation between these functions is provided by the non-renormalization theorem of correlators of BPS operators [15]. Consider the $\frac{1}{2}$ BPS operator tr $\phi^1\phi^2(x)$. Non-renormalization of $\langle \mathrm{tr}\ \phi^1\phi^2(x_1)\ \mathrm{tr}\ \phi^1\phi^2(x_2)\rangle$ implies

$$A'(x_1,x_2) + A(x_1,x_2) + B(x_1,x_2) = 0\,. \qquad (3.4)$$

Another class of Feynman diagrams consists of those that act on three operators. These are shown in figure 2. The corresponding action on $\mathcal{O}[\psi_I]\mathcal{O}[\psi_J]\mathcal{O}[\psi_K]$ is given by

$$\begin{aligned}
\text{Diagram E} &= \delta^{j_m}_{i_l}\delta^{k_n}_{i_{l+1}}\, E(x_1;x_2,x_3)\, N^{-1}\, G(x_1,x_2)\, G(x_1,x_3) \qquad (3.5)\\
\text{Diagram F} &= \left(2\delta^{k_n}_{i_l}\delta^{j_m}_{i_{l+1}} - \delta^{j_m}_{i_l}\delta^{k_n}_{i_{l+1}} - \delta_{i_l i_{l+1}}\delta^{j_m k_n}\right) F(x_1;x_2,x_3)N^{-1}G(x_1,x_2)G(x_1,x_3)\,.
\end{aligned}$$

Again, we can utilize the non-renormalization theorem for three-point BPS operators [21] to constrain $E(x_1;x_2,x_3)$ and $F(x_1;x_2,x_3)$ [15]. The non-renormalization of the three-point correlator $\langle \mathrm{tr}\ \phi^1\phi^2(x_1)\ \mathrm{tr}\ \phi^3\phi^1(x_2)\ \mathrm{tr}\ \phi^2\phi^3(x_3)\rangle$ implies the following relation,

$$\begin{aligned}
&A'(x_1,x_2) + A'(x_2,x_3) + A'(x_1,x_3) + \\
&+2\Big(E(x_1;x_2,x_3) + E(x_2;x_3,x_1) + E(x_3;x_1,x_2) + \\
&\quad +F(x_1;x_2,x_3) + F(x_2;x_3,x_1) + F(x_3;x_1,x_2)\Big) = 0\,. \qquad (3.6)
\end{aligned}$$

[5]The two-point Feynman diagram arising from the potential tr$[\phi^i,\phi^j]^2$ and having three contractions with $\mathcal{O}[\psi_I](x_1)$ and one contraction with $\mathcal{O}[\psi_J](x_2)$ must have a zero net contribution. Besides being quadratically divergent, a non-zero contribution for this diagram would imply perturbative mixing between primary operators of different free scaling dimensions. But such mixing is prohibited by conformal symmetry.

### 3.2 An example: Konishi operator

As a simple example, we work out explicitly the two- and three-point function of the Konishi operator, $K = \frac{1}{\lambda}\ \mathrm{tr}\ \phi^i\phi^i$ . The free two- and three-point diagrams of the Konishi operator are shown in figure 3. At the free theory level, the two-point function is given by

$$\langle K(x_1)K(x_2)\rangle_{\rm free} = \frac{12}{(8\pi^2|x_{12}|^2)^2}\,. \tag{3.7}$$

The $O(\lambda)$ correction gives

$$\begin{aligned}\langle K(x_1)K(x_2)\rangle\Big|_\lambda &= \frac{12}{(8\pi^2|x_{12}|^2)^2}\left[2(A' + A - 5B)\right]\\ &= \frac{12}{(8\pi^2|x_{12}|^2)^2}\left[2(-6B)\right]\\ &= \frac{12}{(8\pi^2|x_{12}|^2)^2}\left\{-12\lambda\frac{1}{16\pi^2}\left[\ln|x_{12}\Lambda|^2 - 1\right]\right\}\end{aligned} \tag{3.8}$$

where in the second line we have used the non-renormalization relation (3.4) and in the third line we have substituted in the expression $B(x_1,x_2) = \frac{\lambda}{16\pi^2}\left[\ln|x_{12}\Lambda|^2 - 1\right]$ (worked out in appendix A using differential regularization). The Konishi operator perturbatively does not mix but has a non-zero anomalous dimension. Comparing (3.7) and (3.8) with (2.3) , we find that the normalization $N_K = 12/(8\pi^2)^2$, $a_K = \frac{3}{8\pi^2}$, and the anomalous dimension $\gamma_K = \frac{3}{4\pi^2}\lambda$. For the three-point function, we have

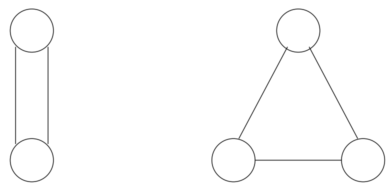



**Figure 3:** Free Feynman diagrams of the Konishi operator.

$$\langle K(x_1)K(x_2)K(x_3)\rangle_{\rm free} = \frac{48}{N\,(8\pi^2)^3|x_{12}x_{13}x_{23}|^2}\,, \tag{3.9}$$

$$\begin{aligned}\langle K(x_1)K(x_2)K(x_3)\rangle\Big|_\lambda &= \frac{48}{N\,(8\pi^2)^3|x_{12}x_{13}x_{23}|^2}\times\\ &\quad\times\Big\{A'(x_1,x_2) + A'(x_2,x_3) + A'(x_1,x_3) +\\ &\quad + 2[E(x_1;x_2,x_3) + E(x_2;x_3,x_1) + E(x_2;x_1,x_2)] +\\ &\quad + 2(2-1-6)[F(x_1;x_2,x_3) + F(x_2;x_3,x_1) + F(x_3;x_1,x_2)]\Big\}\\ &= \frac{48}{N\,(8\pi^2)^3|x_{12}x_{13}x_{23}|^2}\Big\{-12[F(x_1;x_2,x_3) + F(x_2;x_3,x_1) +\\ &\qquad + F(x_3;x_1,x_2)]\Big\}\\ &= \frac{48}{N\,(8\pi^2)^3|x_{12}x_{13}x_{23}|^2}\left(-\frac{3}{4\pi^2}\ln|x_{12}x_{13}x_{23}\Lambda^3|\right)\end{aligned} \tag{3.10}$$

where again, we have used the non-renormalization relation (3.6) and the explicit expression $F(x_3;x_1,x_2) = \frac{\lambda}{32\pi^2}\ln\left|\frac{x_{13}x_{23}\Lambda}{x_{12}}\right|^2$ (from appendix A). Comparing (3.10) with (2.8), we obtain again $\gamma_K = \frac{3}{4\pi^2}\lambda$ and $C = 0$, implying $c^1_{KKK} = -3a_K = -\frac{9}{8\pi^2}$. These values agree with the results for the Konishi operator in [22].

The above Konishi calculation highlights two important features for calculating the one-loop correction, $c^1_{\alpha\beta\rho}$. First, notice that we were able to express any dependence on functions $A'$, $A$, and $E$ in terms of $B$ and $F$ using the non-renormalization relations (3.4) and (3.6). Indeed, in general, only $B$ and $F$ will appear in any two- or three-point function calculation involving scalar operators. Any contribution from $A'$, $A$, and $E$, which contains gauge boson exchange, can always be rewritten in terms of $B$ and $F$. As argued in [20], this must be the case because the functions $A'$, $A$, and $E$ contains terms that are dependent on the gauge fixing parameter. Although the gauge boson exchange Feynman diagrams may have a contribution to the one-loop correction, the functions themselves can not contribute directly to any gauge independent two- or three-point function.

Secondly, we emphasize that though the value of $a_K$ is scheme dependent, $c^1_{KKK}$ is scheme independent. For if instead of using an explicit expression for $B$ and $F$, we consider the scheme independent form $B(x_1,x_2) = b_0 + \frac{\lambda}{16\pi^2}\ln|x_{12}\Lambda|^2$ and $F(x_3;x_1,x_2) = f_0 + \frac{\lambda}{32\pi^2}\ln\left|\frac{x_{13}x_{23}\Lambda}{x_{12}}\right|^2$, leaving the constant $b_0$ and $f_0$ arbitrary. This leads to $c^1_{KKK} = -18(2f_0 - b_0)$. But as pointed out in [16], the expression

$$F(x_3;x_1,x_2) + F(x_1;x_3,x_2) - B(x_1,x_3) = 2f_0 - b_0 \tag{3.11}$$

does not depend on the regulator and therefore is a scheme independent quantity. The precise constant can be found, for example, using differential regularization (as in appendix A) with the result $2f_0 - b_0 = \frac{\lambda}{16\pi^2}$. Thus, in calculating $c^1_{\alpha\beta\rho}$ we have the freedom of working in any scheme. In particular, we can work in the scheme where $f_0 = 0$, as in the Konishi example above, and in fact not consider any three-point Feynman diagrams contributions. As well, we may choose $b_0 = 0$ and not consider any two-point Feynman diagram contributions in calculating $c^1_{\alpha\beta\rho}$. In the next subsection, we will show by applying conformal invariance that $c^1_{\alpha\beta\rho}$ is always proportional to $2f_0 - b_0$.

### 3.3 Formulas for $c^1_{\alpha\beta\rho}$

That the functional dependence of the first order corrections to two- and three-point functions can only take on the form $B$ and $F$ is a powerful supersymmetry constraint. Combining this with constraints from conformal symmetry gives general formulas for $c^1_{\alpha\beta\rho}$ up to operator dependent combinatorial factors. We start with the two point function. The one-loop correction comes only from the $B$ function and implies the general form

$$\begin{aligned}\langle\bar{\mathcal{O}}_\alpha(x_1)\mathcal{O}_\alpha(x_2)\rangle &= \frac{N_\alpha^2}{|x_{12}|^{2\Delta_{0\alpha}}}\left[1 + b_\alpha B(x_1,x_2)\right] \\ &= \frac{N_\alpha^2}{|x_{12}|^{2\Delta_{0\alpha}}}\left[1 + b_\alpha b_0 + \frac{b_\alpha\lambda}{16\pi^2}\ln|x_{12}\Lambda|^2\right]\end{aligned} \tag{3.12}$$

where $b_\alpha$ is an operator dependent combinatorial constant and we have substituted in $B(x_1,x_2) = b_0 + \frac{\lambda}{16\pi^2}\ln|x_{12}\Lambda|^2$. Comparing (3.12) with the required form in (2.3) gives

$$a_\alpha = \frac{b_0}{2}\,b_\alpha\,, \qquad \gamma_\alpha = -\frac{\lambda}{16\pi^2}\,b_\alpha\,, \tag{3.13}$$

This gives the following relationship between $\gamma_\alpha$ and $a_\alpha$,

$$\gamma_\alpha = -\frac{\lambda}{16\pi^2}\left(\frac{2a_\alpha}{b_0}\right) = -\frac{\lambda}{16\pi^2}\,\tilde{a}_\alpha\,, \tag{3.14}$$

where we have defined $\tilde{a}_\alpha = 2a_\alpha/b_0$. From (3.13), calculating the anomalous dimension is just computing the constant $b_\alpha$. An explicit form for this constant in the planar limit will be given in the next section.

Now we proceed with the three-point function calculation. In addition to two-point $B$ functional dependences, we have additional three-point $F$ functional dependences. The first order three-point function can be simply expressed as

$$\begin{aligned}\langle\bar{\mathcal{O}}_\alpha(x_1)\mathcal{O}_\beta(x_2)\mathcal{O}_\rho(x_3)\rangle \;=\; & \frac{C^0}{|x_{12}|^{\Delta_{0\alpha}+\Delta_{0\beta}-\Delta_{0\rho}}|x_{13}|^{\Delta_{0\alpha}+\Delta_{0\rho}-\Delta_{0\beta}}|x_{23}|^{\Delta_{0\beta}+\Delta_{0\rho}-\Delta_{0\alpha}}}\times\\ &\times\Big\{1+[b_{12}B(x_1,x_2)+b_{23}B(x_2,x_3)+b_{31}B(x_3,x_1)]+\\ &\quad+[f^1_{23}F(x_1;x_2,x_3)+f^2_{31}F(x_2;x_3,x_1)+f^3_{12}F(x_3;x_1,x_2)]\Big\}\end{aligned} \tag{3.15}$$

where $f^i_{jk}$ and $b_{ij}$ are again combinatorial constants dependent on the three operators. These constants specify the strength of the associated Feynman diagram contributions. For example, $b_{12}$ is associated with two-point Feynman diagrams that contracts between $\bar{\mathcal{O}}_\alpha$ and $\mathcal{O}_\beta$, and $f^1_{23}$ is associated with three-point Feynman diagram that have two contractions with $\bar{\mathcal{O}}_\alpha$ and one each with $\mathcal{O}_\beta$ and $\mathcal{O}_\rho$. Substituting the expressions for $B(x_1,x_2) = b_0+\frac{\lambda}{16\pi^2}\ln|x_{12}\Lambda|^2$ and $F(x_3;x_1,x_2) = f_0+\frac{\lambda}{32\pi^2}\ln\left|\frac{x_{31}x_{23}\Lambda}{x_{12}}\right|^2$ into (3.15), we obtain

$$\begin{aligned}\langle\bar{\mathcal{O}}_\alpha(x_1)\mathcal{O}_\beta(x_2)\mathcal{O}_\rho(x_3)\rangle \;=\; & \frac{C^0}{|x_{12}|^{\Delta_{0\alpha}+\Delta_{0\beta}-\Delta_{0\rho}}|x_{13}|^{\Delta_{0\alpha}+\Delta_{0\rho}-\Delta_{0\beta}}|x_{23}|^{\Delta_{0\beta}+\Delta_{0\rho}-\Delta_{0\alpha}}}\times\\ &\times\Big\{1+(b_{12}+b_{23}+b_{31})b_0+(f^1_{23}+f^2_{31}+f^3_{12})f_0\\ &\quad+\frac{\lambda}{16\pi^2}\Big[2b_{12}\ln|x_{12}\Lambda|+2b_{23}\ln|x_{23}\Lambda|+2b_{31}\ln|x_{31}\Lambda|\\ &\quad+f^1_{23}\ln\left|\frac{x_{12}x_{31}\Lambda}{x_{23}}\right|+f^2_{31}\ln\left|\frac{x_{23}x_{12}\Lambda}{x_{31}}\right|+f^3_{12}\ln\left|\frac{x_{31}x_{23}\Lambda}{x_{12}}\right|\Big]\Big\}\,.\end{aligned} \tag{3.16}$$

We can compare (3.16) with the expected form for the three point function in (2.8). Together with (3.14), we arrive at the following relations

$$\begin{aligned}\gamma_\alpha &= -\frac{\lambda}{16\pi^2}\left[f^1_{23}+b_{12}+b_{31}\right] = -\frac{\lambda}{16\pi^2}\tilde{a}_\alpha\\ \gamma_\beta &= -\frac{\lambda}{16\pi^2}\left[f^2_{31}+b_{12}+b_{23}\right] = -\frac{\lambda}{16\pi^2}\tilde{a}_\beta\\ \gamma_\rho &= -\frac{\lambda}{16\pi^2}\left[f^3_{12}+b_{31}+b_{23}\right] = -\frac{\lambda}{16\pi^2}\tilde{a}_\rho\,.\end{aligned} \tag{3.17}$$

The above relations for eigen-operators are direct consequences of conformal invariance and supersymmetry. We can rewrite (3.17) as a relationship between two- and three-point



JHEP08(2004)055

constants as follows[6]

$$f^1_{23} = \tilde{a}_\alpha - b_{12} - b_{31}\,, \qquad f^2_{31} = \tilde{a}_\beta - b_{12} - b_{23}\,, \qquad f^3_{12} = \tilde{a}_\rho - b_{31} - b_{23}\,. \tag{3.18}$$

Moreover, from (3.16),

$$\lambda\, C^1 = \lambda\,(c^1_{\alpha\beta\rho} + a_\alpha + a_\beta + a_\rho) = (b_{12} + b_{23} + b_{31})b_0 + (f^1_{23} + f^2_{31} + f^3_{12})f_0\,. \tag{3.19}$$

This implies with (3.17),

$$\begin{aligned}\lambda\, c^1_{\alpha\beta\rho} &= \lambda\left(C^1 - a_\alpha - a_\beta - a_\rho\right)\\ &= b_0(b_{12} + b_{23} + b_{31}) + f_0(f^1_{23} + f^2_{31} + f^3_{12}) -\\ &\quad -\frac{b_0}{2}\left(f^1_{23} + f^2_{31} + f^3_{12} + 2(b_{12} + b_{23} + b_{31})\right)\\ &= \frac{2f_0 - b_0}{2}\left(f^1_{23} + f^2_{31} + f^3_{12}\right).\end{aligned} \tag{3.20}$$

Therefore, we have obtained the general expression for the first order correction to the three-point coefficient and have found that it is indeed proportional to the scheme independent expression, $2f_0 - b_0$. If we now substitute the calculated value of $2f_0 - b_0 = \frac{\lambda}{16\pi^2}$ into (3.20), we arrive at our main result

$$\lambda\, c^1_{\alpha\beta\rho} = \frac{\lambda}{32\pi^2}\left(f^1_{23} + f^2_{31} + f^3_{12}\right), \tag{3.21}$$

which can be equivalently expressed using (3.18) in the form

$$\lambda\, c^1_{\alpha\beta\rho} = -\frac{1}{2}(\gamma_\alpha + \gamma_\beta + \gamma_\rho) - \frac{\lambda}{16\pi^2}(b_{12} + b_{23} + b_{31})\,. \tag{3.22}$$

We have obtained (3.21) and (3.22) without setting either $f_0$ or $b_0$ equal to zero. Hence, in effect, we have summed up all possible Feynman diagrams. But clearly from (3.21), we could have worked in the scheme $(b_0, f_0) = (0, \frac{\lambda}{32\pi^2})$ and just summed up three-point Feynman diagrams. Similarly, summing up only two-point Feynman diagrams in the scheme $(b_0, f_0) = (-\frac{\lambda}{16\pi^2}, 0)$ will give the identical result for $c^1_{\alpha\beta\rho}$ as expressed in (3.22). Thus, these two schemes present two very different prescriptions for calculating $c^1_{\alpha\beta\rho}$.

The formula for $c^1_{\alpha\beta\rho}$ simplify further for the extremal three-point function. Suppose $\Delta_{0\alpha} > \Delta_{0\beta}, \Delta_{0\rho}$, then the three-point function is called extremal if $\Delta_{0\alpha} = \Delta_{0\beta} + \Delta_{0\rho}$. In this special case, there are no contractions between operators $\mathcal{O}_\beta(x_2)$ and $\mathcal{O}_\rho(x_3)$ at the free level. This lack of contractions constrains the types of contributing Feynman diagrams at $O(\lambda)$ and results in $f^2_{31} = f^3_{12} = b_{23} = 0$. The three constraint relations in (3.18) then becomes simply

$$f^1_{23} = \tilde{a}_\alpha - b_{12} - b_{31}\,, \qquad \tilde{a}_\beta = b_{12}\,, \qquad \tilde{a}_\rho = b_{31}\,. \tag{3.23}$$

This implies

$$f^1_{23} = \tilde{a}_\alpha - \tilde{a}_\beta - \tilde{a}_\rho = -\frac{16\pi^2}{\lambda}(\gamma_\alpha - \gamma_\beta - \gamma_\rho) \tag{3.24}$$

[6] Similar equations to (3.18) but with all $\tilde{a}$'s set to zero can be found in [16].



JHEP08(2004)055

where in the last equality, we have used (3.14). Substituting (3.24) into (3.20), we obtain the very simple expression

$$\lambda\, c^1_{\alpha\beta\rho} = \frac{\lambda}{32\pi^2}\, f^1_{23} = \frac{1}{2}(\gamma_\beta + \gamma_\rho - \gamma_\alpha)\,. \tag{3.25}$$

Thus, we have shown that the first order correction to the extremal three-point coefficient depends only on the anomalous dimensions of the three scalar primary operators. This result is valid at finite $N$ and, in particular, applies for operators in the SU(2) subsector.

## 4. $c^1_{\alpha\beta\rho}$ in the planar limit

In the previous section, using constraints from superconformal invariance, we wrote down the general form of $c^1_{\alpha\beta\rho}$ in terms of anomalous dimensions and the combinatorial constants, $b_{ij}$ and $f^i_{jk}$. In general, these constants are not easy to calculate at finite $N$ in the SU(2) subsector. However, at the planar limit, it is possible to write down explicit expressions for these constants in the SO(6) subsector. In this section, we will take the planar limit and consider single trace scalar operators only. We will calculate explicitly $c^1_{123}$ for three generic operators, $\bar{\mathcal{O}}_1[\psi_I](x_1)$, $\mathcal{O}_2[\psi_J](x_2)$, and $\mathcal{O}_3[\psi_K](x_3)$, with length $L_1$, $L_2$, and $L_3$, respectively. Below, we set up our notations by first considering two-point functions and free three-point functions. We then proceed to discuss $c^1_{123}$ for the non-extremal case, $L_1 > L_2 + L_3$, followed by the extremal case, $L_1 = L_2 + L_3$.

### 4.1 Two-point functions and free three-point functions

We write each operator in the form

$$\begin{aligned}\mathcal{O}[\psi_I] &= \frac{1}{\lambda^{L/2}}\,\psi_{i_1\ldots i_L}\,\mathrm{tr}\;\phi^{i_1}\cdots\phi^{i_L}\\ &= \frac{1}{\lambda^{L/2}L}\tilde{\psi}_{i_1\ldots i_L}\,\mathrm{tr}\,\phi^{i_1}\cdots\phi^{i_L}\,,\end{aligned} \tag{4.1}$$

where in the second line we have introduced a normalized coefficient $\tilde{\psi} = L\psi$. We emphasize that all indices in $I = \{i_1,\ldots,i_L\}$ are summed over in (4.1), and throughout, we will not explicitly write out the summation symbol. Also, the presence of the trace implies that $\psi_{i_1\ldots i_L}$ must be invariant under cyclic permutation of the indices. Thus, for example,

$$\begin{aligned}\mathrm{tr}\,\phi^1\phi^2\phi^3 &= \psi_{i_1i_2i_3}\,\mathrm{tr}\,\phi^{i_1}\phi^{i_2}\phi^{i_3}\\ &= \psi_{123}\;\mathrm{tr}\,\phi^1\phi^2\phi^3 + \psi_{231}\,\mathrm{tr}\,\phi^2\phi^3\phi^1 + \psi_{312}\,\mathrm{tr}\,\phi^3\phi^1\phi^2\\ &= \frac{1}{3}\left(\tilde{\psi}_{123}\;\mathrm{tr}\,\phi^1\phi^2\phi^3 + \tilde{\psi}_{231}\,\mathrm{tr}\,\phi^2\phi^3\phi^1 + \tilde{\psi}_{312}\,\mathrm{tr}\,\phi^3\phi^1\phi^2\right)\end{aligned} \tag{4.2}$$

with $\psi_{123} = \psi_{231} = \psi_{312} = 1/3$ and $\tilde{\psi}_{123} = \tilde{\psi}_{231} = \tilde{\psi}_{312} = 1$. In writing out the form of the two and three-point functions, it is useful to establish conventions for labeling the indices of the $\psi$ coefficients as shown in figure 4. The two-point function at one-loop can then be expressed as

$$\langle\bar{\mathcal{O}}_\alpha[\psi_{I'}](x_1)\mathcal{O}_\alpha[\psi_I](x_2)\rangle = \left\langle \frac{1}{\lambda^{L/2}}\psi_{i'_L\ldots i'_1}\;\mathrm{tr}\,\phi^{i'_L}\cdots\phi^{i'_1}(x_1)\frac{1}{\lambda^{L/2}}\psi_{i_1\ldots i_L}\;\mathrm{tr}\,\phi^{i_1}\cdots\phi^{i_L}(x_2)\right\rangle$$

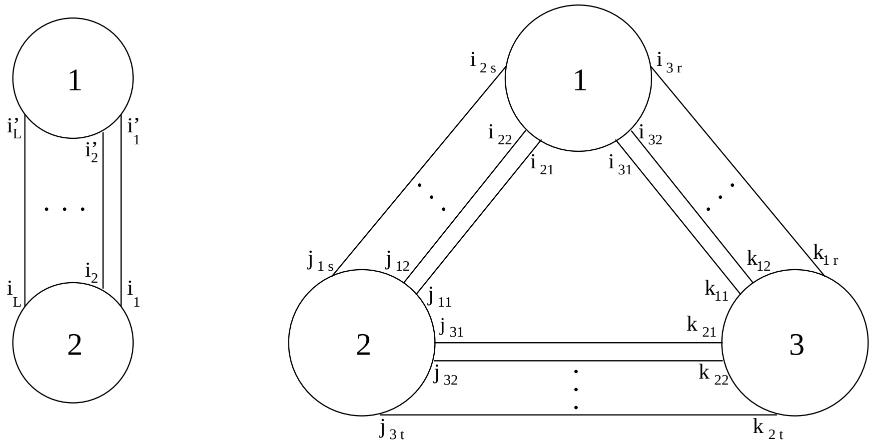



**Figure 4:** Labeling of the free two- and three-point operators.

$$= \frac{\tilde{\psi}_{I'}\psi_I \mathcal{I}_2 + \tilde{\psi}_{I'}\tilde{\psi}_I (2\mathcal{P} - 2\mathcal{I} - \mathcal{K})_{i_1 i_2}^{i'_1 i'_2} \mathcal{I}'_2 B(x_1, x_2)}{[8\pi^2 |x_{12}|^2]^L} \tag{4.3}$$

where

$$\mathcal{P}_{ij}^{kl} = \delta_i^l \delta_j^k\,, \qquad \mathcal{I}_{ij}^{kl} = \delta_i^k \delta_j^l\,, \qquad \mathcal{K}_{ij}^{kl} = \delta_{ij}\,\delta^{kl}\,. \tag{4.4}$$

$$\mathcal{I}_2 = \delta_{i_1}^{i'_1} \delta_{i_2}^{i'_2} \ldots \delta_{i_L}^{i'_L}, \qquad \mathcal{I}'_2 = \delta_{i_3}^{i'_3} \ldots \delta_{i_L}^{i'_L}\,. \tag{4.5}$$

The ordering of the indices in line one of (4.3) follows from figure 4, using a "right-handed" counter-clockwise convention. The ordering for $\psi_{I'}$ is reversed to take account of hermitian conjugation. In the second line of (4.3), we have summed up all contributing two-point Feynman diagrams. $\mathcal{P}$, $\mathcal{I}$, and $\mathcal{K}$ are the permutation, identity, and trace operators typical of nearest neighbor spin chain interactions. $\mathcal{I}_2$ represents full free contractions of the indices of the two operators and the primed $\mathcal{I}'_2$, written explicitly in (4.5), represents free contractions of all indices that have not yet been contracted. Substituting $B(x_1, x_2) = b_0 + \frac{\lambda}{16\pi^2} \ln |x_{12}\Lambda|^2$ into (4.3) and comparing with the expected form for two-point functions in (3.12), we find for the anomalous dimension (3.14),

$$\gamma_\alpha = -\frac{\lambda}{16\pi^2} \frac{\tilde{\psi}_{I'}\tilde{\psi}_I (2\mathcal{P} - 2\mathcal{I} - \mathcal{K})_{i_1 i_2}^{i'_1 i'_2} \mathcal{I}'_2}{\tilde{\psi}_{I'}\psi_I \mathcal{I}} = -\frac{\lambda}{16\pi^2} \tilde{a}_\alpha\,. \tag{4.6}$$

Recalling that $\mathcal{O}_\alpha[\psi_I]$ is an eigen-operator and also that the indices of $\psi$ are cyclically invariant, (4.6) is indeed identical to the anomalous dimension formula given in [2].

For the three-point function, the $\psi$ indices are labeled as follows,

$$\begin{aligned} \bar{\mathcal{O}}_1[\psi_I]:&\ \ \psi_I = \psi_{i_{31} i_{32} \ldots i_{3r} i_{2s} \ldots i_{22} i_{21}} \\ \mathcal{O}_2[\psi_J]:&\ \ \psi_J = \psi_{j_{11} j_{12} \ldots j_{1s} j_{3t} \ldots j_{32} j_{31}} \\ \mathcal{O}_3[\psi_K]:&\ \ \psi_K = \psi_{k_{21} k_{22} \ldots k_{2t} k_{1r} \ldots k_{12} k_{11}} \end{aligned} \tag{4.7}$$

where for example $i_{32}$ signifies the "2nd" contraction starting from $I$ to the "3rd" operator. Note that $r$, $s$, and $t$ are the number of contractions between operators 3-1, 1-2, and 2-3 respectively. These three numbers are uniquely determined by the lengths of the operators by the following relations

$$r = \frac{1}{2}\left(L_3 + L_1 - L_2\right), \qquad s = \frac{1}{2}\left(L_1 + L_2 - L_3\right), \qquad t = \frac{1}{2}\left(L_2 + L_3 - L_1\right). \tag{4.8}$$

Moreover, if $r$, $s$, and $t$ as defined in (4.8) are not all integers, then the three-point function must be zero. For at the free level, the full contraction is given by

$$\mathcal{I}_3 = \delta^{k_{11}}_{i_{31}} \delta^{k_{12}}_{i_{32}} \cdots \delta^{k_{1r}}_{i_{3r}} \, \delta^{j_{11}}_{i_{21}} \delta^{j_{12}}_{i_{22}} \cdots \delta^{j_{1s}}_{i_{2s}} \, \delta^{k_{21}}_{j_{31}} \delta^{k_{22}}_{j_{32}} \cdots \delta^{k_{2t}}_{j_{3t}} \,. \tag{4.9}$$

We can use $\mathcal{I}_3$ to simply express the free three-point function as

$$\langle \bar{\mathcal{O}}_1[\psi_I](x_1) \mathcal{O}_2[\psi_J](x_2) \mathcal{O}_3[\psi_K](x_3)\rangle_{\text{free}} = \frac{\tilde{\psi}_I \tilde{\psi}_J \tilde{\psi}_K \mathcal{I}_3}{N\,(8\pi^2)^{r+s+t} |x_{31}|^{2r} |x_{12}|^{2s} |x_{23}|^{2t}} \,. \tag{4.10}$$

And as a contribution to the operator production expansion (OPE), (4.10) implies

$$\mathcal{O}_2[\psi_J](x_2)\mathcal{O}_3[\psi_k](x_3) \sim \frac{\tilde{\psi}_J \tilde{\psi}_K \delta^{k_{21}}_{j_{31}} \delta^{k_{22}}_{j_{32}} \cdots \delta^{k_{2t}}_{j_{3t}}}{N(8\pi^2)^t |x_{23}|^{2t}} \operatorname{tr} \phi^{j_{11}} \phi^{j_{12}} \cdots \phi^{j_{1s}} \phi^{k_{1r}} \cdots \phi^{k_{12}} \phi^{k_{1r}}(x_3) + \cdots \tag{4.11}$$

with the value of $t$ fixing the length of the resulting operator.

## 4.2 Non-extremal three-point function

The expression for $c^1_{123}$ can be expressed explicitly in the planar limit. Contribution to $c^1_{123}$ consists of three families of two-point Feynman diagrams and six different E and F types three-point Feynman diagrams. Some of which are shown in figure 5.

Again, the nonzero contribution to $c^1_{123}$ consists only of two-point interaction with $B$ functional dependence and three-point interaction with $F$ functional dependence. The one-loop order three-point function is thus expressed as

$$\begin{aligned}
&\langle \bar{\mathcal{O}}_1[\psi_I](x_1) \mathcal{O}_2[\psi_J](x_2) \mathcal{O}_3[\psi_K](x_3)\rangle = \\
&\quad = \frac{\tilde{C}^0}{N\,(8\pi^2)^{r+s+t} |x_{31}|^{2r} |x_{12}|^{2s} |x_{23}|^{2t}} \times \\
&\quad\quad \times \Big\{ 1 + \left[b_{12} B(x_1, x_2) + b_{23} B(x_2, x_3) + b_{31} B(x_3, x_1)\right] + \\
&\quad\quad\quad + \left[f^1_{23} F(x_1; x_2, x_3) + f^2_{31} F(x_2; x_3, x_1) + f^3_{12} F(x_3; x_1, x_2)\right] \Big\}
\end{aligned} \tag{4.12}$$

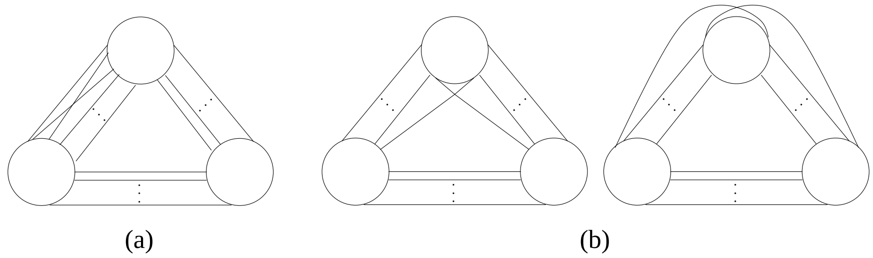



**Figure 5:** Examples of planar diagrams contributing to $c^1_{123}$. $(a)$ A two-point B type diagram. $(b)$ Two three-point F type diagrams.

where we have defined $\tilde{C}^0 = C^0 N(8\pi^2)^{r+s+t} = \tilde{\psi}_I \tilde{\psi}_J \tilde{\psi}_K \mathcal{I}_3$. Summing up all contributions, we find the following expressions for the constants

$$
\begin{aligned}
b_{12} &= \sum_{l=1}^{s-1} (2\mathcal{P} - 2\mathcal{I} - \mathcal{K})^{j_{1\,l} j_{1\,l+1}}_{i_{2\,l} i_{2\,l+1}} \frac{\tilde{\psi}_I \tilde{\psi}_J \tilde{\psi}_K \mathcal{I}'_3}{\tilde{C}^0} \\
b_{23} &= \sum_{l=1}^{t-1} (2\mathcal{P} - 2\mathcal{I} - \mathcal{K})^{k_{2\,l} k_{2\,l+1}}_{j_{3\,l} j_{3\,l+1}} \frac{\tilde{\psi}_I \tilde{\psi}_J \tilde{\psi}_K \mathcal{I}'_3}{\tilde{C}^0} \\
b_{31} &= \sum_{l=1}^{r-1} (2\mathcal{P} - 2\mathcal{I} - \mathcal{K})^{i_{3\,l} i_{3\,l+1}}_{k_{1\,l} k_{1\,l+1}} \frac{\tilde{\psi}_I \tilde{\psi}_J \tilde{\psi}_K \mathcal{I}'_3}{\tilde{C}^0} \\
f^1_{23} &= \left[ (2\mathcal{P} - 2\mathcal{I} - \mathcal{K})^{j_{11} k_{11}}_{i_{21} i_{31}} \mathcal{I}'_3 + (2\mathcal{P} - 2\mathcal{I} - \mathcal{K})^{k_{1r} j_{1s}}_{i_{3r} i_{2s}} \mathcal{I}'_3 \right] \frac{\tilde{\psi}_I \tilde{\psi}_J \tilde{\psi}_K}{\tilde{C}^0} \\
f^2_{31} &= \left[ (2\mathcal{P} - 2\mathcal{I} - \mathcal{K})^{k_{21} i_{21}}_{j_{31} j_{11}} \mathcal{I}'_3 + (2\mathcal{P} - 2\mathcal{I} - \mathcal{K})^{i_{2s} k_{2t}}_{j_{1s} j_{3t}} \mathcal{I}'_3 \right] \frac{\tilde{\psi}_I \tilde{\psi}_J \tilde{\psi}_K}{\tilde{C}^0} \\
f^3_{12} &= \left[ (2\mathcal{P} - 2\mathcal{I} - \mathcal{K})^{i_{31} j_{31}}_{k_{11} k_{21}} \mathcal{I}'_3 + (2\mathcal{P} - 2\mathcal{I} - \mathcal{K})^{j_{3t} i_{3r}}_{k_{2t} k_{1r}} \mathcal{I}'_3 \right] \frac{\tilde{\psi}_I \tilde{\psi}_J \tilde{\psi}_K}{\tilde{C}^0} \,.
\end{aligned} \tag{4.13}
$$

In above, $\mathcal{I}'_3$ is defined to be the required free contractions that fully contract the remaining indices and $\mathcal{I}_3$ is the full free contraction defined in (4.9). Thus, for example, with all the contractions explicitly written out,

$$
\begin{aligned}
f^1_{23} = \frac{\tilde{\psi}_I \tilde{\psi}_J \tilde{\psi}_K}{\tilde{C}^0} & \left[ (2\mathcal{P} - 2\mathcal{I} - \mathcal{K})^{j_{11} k_{11}}_{i_{21} i_{31}} \mathcal{I}^{k_{1r} j_{1s}}_{i_{3r} i_{2s}} + (2\mathcal{P} - 2\mathcal{I} - \mathcal{K})^{k_{1r} j_{1s}}_{i_{3r} i_{2s}} \mathcal{I}^{j_{11} k_{11}}_{i_{21} i_{31}} \right] \times \\
& \times \delta^{j_{12}}_{i_{22}} \delta^{j_{13}}_{i_{23}} \cdots \delta^{j_{1\,s-1}}_{i_{2\,s-1}} \delta^{k_{12}}_{i_{32}} \delta^{k_{13}}_{i_{33}} \cdots \delta^{k_{1\,r-1}}_{i_{3\,r-1}} \delta^{k_{21}}_{j_{31}} \delta^{k_{22}}_{j_{32}} \cdots \delta^{k_{2\,t}}_{j_{3\,t}}
\end{aligned} \tag{4.14}
$$

where $\mathcal{I}^{kl}_{ij} = \delta^k_i \delta^l_j$. The constants in (4.13) together with either (3.21) or (3.22) gives the explicit formula for $c^1_{123}$. In the next section, we will interpret the planar expressions from the spin chain perspective.

### 4.3 Extremal three-point function

For the extremal three-point function, with $L_1 = L_2 + L_3$, the number of contractions between operators 2 and 3 is zero (i.e. $t = 0$). We have found that $c^1_{123}$ is simply given by (3.25) in terms of the anomalous dimensions of the three scalar primary operators. Nevertheless, it is interesting to write down the explicit expressions for the three nonzero constants — $f^1_{23}$, $b_{12}$, and $b_{31}$ — in the planar limit. Interestingly, they are not those in (4.13). For in the extremal case, two subtleties must be incorporated into the calculation. First, at the free level, the mixing of $\mathcal{O}_1[\psi_I](x_1)$ with the double trace operator $\mathcal{O}'_1(x_1) = \mathcal{O}_2[\psi_J]\mathcal{O}_3[\psi_K](x_1)$ must be taken into account [23]. That the mixing coefficient contains a factor of $N^{-1}$ is compensated by the fact that the free three-point function of $\mathcal{O}'_1(x_1)$ with $\mathcal{O}_2(x_2)$ and $\mathcal{O}_3(x_3)$ is $O(N^0)$ instead of $O(N^{-1})$. Though $\mathcal{O}'_1$ makes no contribution to first order correction $C^1$, $C^0$ can not longer be expressed simply as in (4.10). Second, in addition to the Feynman diagrams in figure 5, there are planar non-nearest neighbor diagrams that also must be considered. The new diagrams, E' and F', are shown in figure 6. The contributions of diagrams E' exactly cancel out those of diagrams E of figure 2. However,

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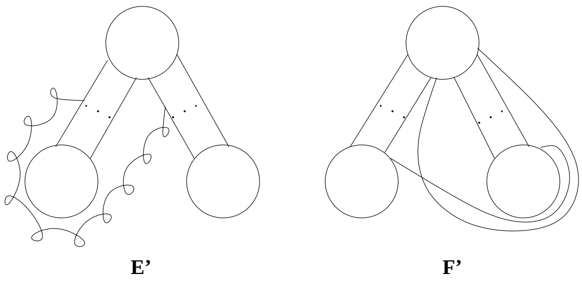


**Figure 6:** Non-nearest neighbor diagram for extremal three-point functions.

diagram F' gives a nonzero non-nearest neighbor contribution not present in the non-extremal case. Adding up the contributing Feynman diagrams, we find

$$\begin{aligned}
f^1_{23} &= \left[(2\mathcal{P}-\mathcal{I}-\mathcal{K})^{j_{11}k_{11}}_{i_{21}i_{31}}\mathcal{I}'_3 + (2\mathcal{P}-\mathcal{I}-\mathcal{K})^{k_{1r}j_{1s}}_{i_{3r}i_{2s}}\mathcal{I}'_3 + (2\mathcal{P}-\mathcal{I}-\mathcal{K})^{i_{3r}k_{1r}}_{j_{11}i_{21}}\mathcal{I}'_3\right]\frac{\tilde\psi_I\tilde\psi_J\tilde\psi_K}{\tilde C^0}\\
b_{12} &= \left[\sum_{l=1}^{s-1}(2\mathcal{P}-2\mathcal{I}-\mathcal{K})^{j_{1\,l}j_{1\,l+1}}_{i_{2\,l}i_{2\,l+1}}\mathcal{I}'_3 + (2\mathcal{P}-2\mathcal{I}-\mathcal{K})^{j_{1\,s}j_{11}}_{i_{2\,s}i_{21}}\mathcal{I}'_3\right]\frac{\tilde\psi_I\tilde\psi_J\tilde\psi_K}{\tilde C^0}\\
b_{31} &= \left[\sum_{l=1}^{r-1}(2\mathcal{P}-2\mathcal{I}-\mathcal{K})^{i_{3\,l}i_{3\,l+1}}_{k_{1\,l}k_{1\,l+1}}\mathcal{I}'_3 + (2\mathcal{P}-2\mathcal{I}-\mathcal{K})^{k_{1\,r}k_{11}}_{i_{3\,r}i_{31}}\mathcal{I}'_3\right]\frac{\tilde\psi_I\tilde\psi_J\tilde\psi_K}{\tilde C^0}
\end{aligned}\tag{4.15}$$

where once again, $\mathcal{I}'_3$ denotes the required free contractions to fully contract each term. In $f^1_{23}$, the third term corresponds to the contribution from the non-nearest neighbor F' type Feynman diagrams in figure 6. The expressions in (4.15) must satisfy (3.23), which also gives an indirect way of calculating $\tilde C^0$.

## 5. Planar one-loop correction and integrable spin chains

The general formulas for $c^1_{123}$ in the previous section are given in explicit forms in the planar limit. In light of the connection of operators with states in SO(6) integrable spin chain in the planar limit [2], we will interpret $c^1_{123}$ from the spin chain perspective.

### 5.1 Mapping SYM into spin chain

As is clear from the one-loop planar formula in (4.6), the anomalous dimension is just a combinatorial factor depending only on the SO(6) chain of indices of the operator and nothing else. Therefore, focusing only on the SO(6) indices, we can naturally map $\mathcal{O}[\psi_I]$ to a spin chain state

$$|\Psi\rangle = \tilde\psi_{i_1\ldots i_L}|i_1\ldots i_L\rangle \tag{5.1}$$

where, as before, $\tilde\psi_{i_1\ldots i_L}$ is invariant under cyclic permutation. Note that $|i_1\ldots i_L\rangle$ effectively spans the Hilbert space $\mathcal{H}=V^{\otimes L}$ where $V=\mathbb{R}^6$. We will define the conjugate of $|\Psi\rangle$ as[7]

$$\langle\Psi| = \langle i_1\ldots i_L|\tilde\psi_{i_L\ldots i_1} \tag{5.2}$$

[7]In the SO(6) subsector, eigenstates of the planar anomalous dimension matrix can be taken to be real vectors, since that matrix is real symmetric. Therefore, we can impose that the coefficients $\tilde\psi_{i_1\ldots i_L}$ in (5.1) are real numbers.

where we have reversed the order of the indices so that the inner product is given by $\langle a_1 \ldots a_r | b_1 \ldots b_r \rangle = \delta^{a_1}_{b_1} \ldots \delta^{a_r}_{b_r}$. As pointed out by Minahan and Zarembo [2], the planar one-loop anomalous dimension matrix is just proportional to the hamiltonian of the SO(6) integrable closed spin chain. In our notations, the anomalous dimension (4.6) can be written simply as

$$\gamma = \frac{\lambda}{16\pi^2} \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle} \tag{5.3}$$

where $|\Psi\rangle$ is assumed to be an eigenstate of the SO(6) integrable closed hamiltonian,

$$H = \sum_{l=1}^{L} \left( \mathcal{K}_{l,l+1} + 2\mathcal{I}_{l,l+1} - 2\mathcal{P}_{l,l+1} \right). \tag{5.4}$$

Being integrable, the eigenvalues and eigenstates of $H$ can be found using the algebraic Bethe ansatz techniques.

Now, for the one-loop correction $c^1_{123}$, the explicit planar expressions for both non-extremal and extremal cases also only depend on the SO(6) indices of the three eigen-operators. (See eqs. (3.21), (3.25), and (4.13).) This again implies that there is an SO(6) spin chain interpretation. For the extremal case, the integrable structure is self-evident since, as in (3.25), $c^1_{123}$ is just a linear combination of anomalous dimensions. For the non-extremal case, we will show the integrability in the spin chain description using the two-point interaction prescription of (3.22).

In the explicit expressions for $b_{12}, b_{23}$, and $b_{31}$ in (4.13), the action on the SO(6) indices is similar to (5.4) except that there is no interaction between sites at the two ends. For example, for $b_{31}$, sites $k_{1r}$ and $k_{11}$ do not interact with each other. A hamiltonian such as

$$H_r = \sum_{l=1}^{r-1} (\mathcal{K}_{l,l+1} + 2\mathcal{I}_{l,l+1} - 2\mathcal{P}_{l,l+1}) : V^{\otimes r} \to V^{\otimes r} \tag{5.5}$$

where the two ends do not interact is called an open spin chain hamiltonian.[8] An integrable open spin chain is described by an integrable closed spin chain interaction in the bulk together with boundary conditions at the two ends denoted by $K^{\pm}$. In order for the resulting open chain to be integrable, $K^{\pm}$ must satisfy the boundary Yang-Baxter equations [27]

$$\begin{aligned} R_{12}(u-v)K^-_1(u)R_{12}(u+v)K^-_2(v) &= K^-_2(v)R_{12}(u+v)K^-_1(u)R_{12}(u-v) \times \\ &\quad \times R_{12}(-u+v)K^{+t_1}_1(u)R_{12}(-u-v-2\eta)K^{+t_2}_2(v) \\ &= K^{+t_2}_2(v)R_{12}(-u-v-2\eta)K^{+t_1}_1(u)R_{12}(-u+v) \end{aligned} \tag{5.6}$$

where for SO(6) spin chain, $\eta = -2$ and the $R$-matrix is given by

$$R_{12}(u) = \frac{1}{2}\left[ u(u-2)\mathcal{I}_{12} - (u-2)\mathcal{P}_{12} + u\mathcal{K}_{12} \right]. \tag{5.7}$$

$t_i$ $(i = 1, 2)$ in (5.6) denotes the transpose on the $i^{\text{th}}$ vector space. The hamiltonian in (5.5) corresponds to the "free" boundary conditions $K^{\pm} = 1$ which trivially satisfies (5.6).

[8]See [24, 25, 26] for recent works on the open spin chain description of anomalous dimensions in certain gauge theories.

Having identified open spin chain hamiltonians in the definitions of $b_{ij}$, one may wonder how open spin chain states can arise from closed spin chain states. Simply, we can split a closed spin chain into two, thereby breaking the periodicity, to give two associated open spin chains. In this respect, operators $\mathcal{O}_{1,2,3}$ can be regarded as matrix operators acting on the open spin chain Hilbert space as follows. Recalling that the lengths of operators and $r, s, t$ in (4.8) are related by

$$L_1 = r + s\,, \qquad L_2 = s + t\,, \qquad L_3 = t + r\,, \tag{5.8}$$

we define the open spin chain matrix operators $\Psi_{1,2,3}$ corresponding to operators $\mathcal{O}_{1,2,3}$ by

$$\begin{aligned} \Psi_1 &= \tilde{\psi}_{i_{31}\ldots i_{3r} i_{2s}\ldots i_{21}} |i_{31}\ldots i_{3r}\rangle\langle i_{21}\ldots i_{2s}| \quad : V^{\otimes s} \to V^{\otimes r} \\ \Psi_2 &= \tilde{\psi}_{j_{11}\ldots j_{1s} j_{3t}\ldots j_{31}} |j_{11}\ldots j_{1s}\rangle\langle j_{31}\ldots j_{3t}| \quad : V^{\otimes t} \to V^{\otimes s} \\ \Psi_3 &= \tilde{\psi}_{k_{21}\ldots k_{2t} k_{1r}\ldots k_{11}} |k_{21}\ldots k_{2t}\rangle\langle k_{11}\ldots k_{1r}| \quad : V^{\otimes r} \to V^{\otimes t}\,. \end{aligned} \tag{5.9}$$

The ket $|i_{31}\ldots i_{3r}\rangle$ and the bra $\langle i_{21}\ldots i_{2s}|$ represent the open spin chain states associated with the closed spin chain $\mathcal{O}_1$. In terms of these spin chain operators, $c^1_{123}$, as given by (3.22) with (4.13), is written simply as

$$\lambda c^1_{123} = -\frac{1}{2}\sum_{i=1}^{3}\gamma_i + \frac{\lambda}{16\pi^2}\frac{\mathrm{Tr}_r(H_r\Psi_1\Psi_2\Psi_3 + \Psi_1 H_s\Psi_2\Psi_3 + \Psi_1\Psi_2 H_t\Psi_3)}{\mathrm{Tr}_r\,\Psi_1\Psi_2\Psi_3}\,. \tag{5.10}$$

Above, $\mathrm{Tr}_r$ is the trace over the vector space $V^{\otimes r}$ and $H_r$ is the hamiltonian acting on a open spin chain of length $r$ as in (5.5).

Equation (5.10) can be further simplified by introducing the following matrices $M_{r,s,t}$

$$M_r = \frac{\Psi_1\Psi_2\Psi_3}{\mathrm{Tr}_r\,\Psi_1\Psi_2\Psi_3}\,, \qquad M_s = \frac{\Psi_2\Psi_3\Psi_1}{\mathrm{Tr}_s\,\Psi_2\Psi_3\Psi_1}\,, \qquad M_t = \frac{\Psi_3\Psi_1\Psi_2}{\mathrm{Tr}_t\,\Psi_3\Psi_1\Psi_2}\,. \tag{5.11}$$

In terms of these matrices, (5.10) is rewritten as

$$\begin{aligned} \lambda c^1_{123} &= -\frac{1}{2}\sum_{i=1}^{3}\gamma_i + \frac{\lambda}{16\pi^2}\sum_{k=r,s,t}\mathrm{Tr}_k(H_k M_k) \\ &= -\frac{1}{2}\sum_{i=1}^{3}\gamma_i + \frac{\lambda}{16\pi^2}\sum_{k=r,s,t}\mathrm{Tr}_k(H_k \rho_k) \end{aligned} \tag{5.12}$$

where $\rho_k$ is the symmetric part of $M_k$

$$\rho_k = \frac{1}{2}(M_k + M_k^T)\,, \tag{5.13}$$

and $M_k^T$ denotes the transpose of $M_k$. Note that from (5.11) and (5.13), $\rho_k$ satisfies

$$\mathrm{Tr}_k(\rho_k) = 1\,. \tag{5.14}$$

From this property and the fact that $\rho_k$ is a real symmetric matrix (see footnote 7), we interpret $\rho_k$ as a "density matrix" and the second term of $c^1_{123}$ in (5.12) as the ensemble



JHEP08(2004)055

average of open chain hamiltonian $H_k$ with weight $\rho_k$. The appearance of density matrices seems natural from our definition of open chain states. By the construction of open spin chain states as the two segments of one closed spin chain, the resulting open spin states are correlated, and in some sense, they can be thought of as "mixed states."

The integrability of open spin chain hamiltonian $H_k$ could be useful to compute $c^1_{123}$ given in (5.12). By utilizing the powerful techniques of integrable system, such as the algebraic Bethe ansatz, we can in principle diagonalize $H_k$ as

$$H_k = \sum_{\alpha=1}^{6^k} \mathcal{E}^{(k)}_\alpha |v^{(k)}_\alpha\rangle\langle v^{(k)}_\alpha| \,. \tag{5.15}$$

Here, we orthonormalized the eigenvectors: $\langle v^{(k)}_\alpha | v^{(k)}_\beta\rangle = \delta_{\alpha\beta}$. Once the eigenvalues and eigenvectors of $H_k$ are known, $\mathrm{Tr}_k(H_k\rho_k)$ appearing in (5.12) can be written as the average of eigenvalues with respect to the "probability" $p^{(k)}_\alpha = \langle v^{(k)}_\alpha|\rho_k|v^{(k)}_\alpha\rangle$

$$\mathrm{Tr}_k(H_k\rho_k) = \sum_{\alpha=1}^{6^k} \mathcal{E}^{(k)}_\alpha p^{(k)}_\alpha \,, \tag{5.16}$$

where $\sum_\alpha p^{(k)}_\alpha = 1$.

In our spin chain interpretation of $c^1_{123}$, we have utilized the connection between $b_{ij}$ and open spin chain hamiltonians. In contrast, for $f^i_{jk}$, its SO(6) index structure as given in (4.13) does not directly lend itself to an interpretation in terms of conventional spin chain language. We can however use the relations between two- and three-point constants in (3.18) to obtain a heuristic understanding of $f^i_{jk}$. For instance, (3.18) suggests that we can write $f^1_{23}$ as the energy difference between a closed spin chain and its associated open spin chains

$$f^1_{23} = -\frac{\langle\Psi_1|H|\Psi_1\rangle}{\langle\Psi_1|\Psi_1\rangle} + \mathrm{Tr}_r(H_r\rho_r) + \mathrm{Tr}_s(H_s\rho_s) \,. \tag{5.17}$$

In other words, $f^1_{23}$ represents roughly the "energy cost" in splitting a closed chain of length $L_1 = r + s$ into the associated open chains of lengths $r$ and $s$. Since $c^1_{123} = \frac{1}{32\pi^2}\left(f^1_{23} + f^2_{31} + f^3_{12}\right)$, $c^1_{123}$ as given in (5.12) can also be interpreted as the total energy cost in splitting all three closed chains. As for the extremal case where $\lambda c^1_{123} = \frac{1}{2}(\gamma_2 + \gamma_3 - \gamma_1)$, $c^1_{123}$ has a clear interpretation as the energy cost in the splitting of a single closed chain into two separate closed chains.

In passing, we note that the two terms in $\rho_k$ (5.13) have an interesting interpretation in the diagrammatic representation of $c^1_{123}$, as shown in figure 7. In writing operators $\mathcal{O}_{1,2,3}$ as matrix operators $\Psi_{1,2,3}$, we implicitly assumed that the diagram describing the index contraction has a definite orientation, which corresponds to the first term $M_k$ in (5.13). On the other hand, the diagram for $M^T_k$ carries the opposite orientation from that of $M_k$. Equations (5.12) and (5.13) imply that the two orientations contribute to $c^1_{123}$ with equal weight. This is consistent with the fact that the three-point function coefficient $c_{123}$ is symmetric with respect to the indices $1, 2, 3$.

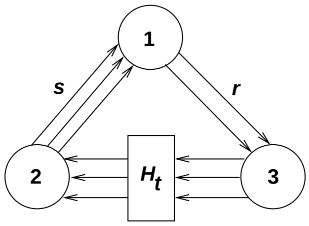


**Figure 7:** Schematic diagram depicting one of the terms in $c^1_{123}$. $H_t$ is the open spin chain hamiltonian. (In this figure $r = 2$, $s = t = 3$.) The arrows indicate the orientation we choose in order to write $\mathcal{O}_{1,2,3}$ as matrix operators $\Psi_{1,2,3}$. $c^1_{123}$ is clearly independent of the choice of orientation.

### 5.2 Some examples

Let us illustrate our general formula (5.10) with some examples of operators with small scaling dimensions. We list the eigen-operators consisting up to three scalar fields,

$$\begin{aligned}
K &= \frac{1}{\lambda}\,\mathrm{tr}\,\phi^m\phi^m \\
\mathcal{O}_{(ij)} &= \frac{1}{\lambda}(\mathrm{tr}\,\phi^i\phi^j - \delta^{ij}\frac{1}{6}\,\mathrm{tr}\,\phi^m\phi^m) \\
K_i &= \frac{1}{\lambda^{3/2}}\,\mathrm{tr}\,\phi^m\phi^m\phi^i \\
\mathcal{O}_{(123)} &= \frac{1}{\lambda^{3/2}}(\mathrm{tr}\,\phi^1\phi^2\phi^3 + \mathrm{tr}\,\phi^3\phi^2\phi^1) \\
\mathcal{O}_{[123]} &= \frac{1}{\lambda^{3/2}}(\mathrm{tr}\,\phi^1\phi^2\phi^3 - \mathrm{tr}\,\phi^3\phi^2\phi^1)\,.
\end{aligned} \tag{5.18}$$

$\mathcal{O}_{(123)}$ belongs to the symmetric traceless representation of SO(6) and $\mathcal{O}_{[123]}$ belongs to the anti-symmetric representation. Their anomalous dimensions are given by

$$\gamma_{\mathcal{O}_{(ij)}} = \gamma_{\mathcal{O}_{(123)}} = 0\,, \qquad \gamma_K = \gamma_{\mathcal{O}_{[123]}} = \frac{3}{4\pi^2}\lambda\,, \qquad \gamma_{K_i} = \frac{1}{2\pi^2}\lambda. \tag{5.19}$$

And note that $\mathcal{O}_{(ij)}$ and $\mathcal{O}_{(123)}$ are 1/2 BPS operators.

As a first example, let us reconsider $c^1_{KKK}$ (studied in section 3) from the general formula (5.10). The matrix associated with the Konishi operator is $\Psi_K = 2\sum_{m=1}^{6}|m\rangle\langle m|$. Since the open spin hamiltonian $H_1$ acting on length 1 chain is zero, there is no contribution from the second term in (5.10). Therefore, $\lambda c^1_{KKK}$ is given by $-\frac{3}{2}\gamma_K = -\frac{9}{8\pi^2}\lambda$, in agreement with our computation in section 3. Next, let us consider the slightly more complicated example $c^1_{\mathcal{O}_{(12)}K_1K_2}$. The matrices associated with operators $\mathcal{O}_{(12)}$, $K_1$, and $K_2$ are given by

$$\begin{aligned}
\Psi_{\mathcal{O}_{12}} &= |1\rangle\langle 2| + |2\rangle\langle 1| \\
\Psi_{K_1} &= \sum_{m=1}^{6} |1\rangle\langle m,m| + |m\rangle\langle m,1| + |m\rangle\langle 1,m| \\
\Psi_{K_2} &= \sum_{m=1}^{6} |m,m\rangle\langle 2| + |m,2\rangle\langle m| + |2,m\rangle\langle m|\,.
\end{aligned} \tag{5.20}$$

JHEP08(2004)055

Again, the open spin chain hamiltonian acting on the length 1 part is zero. The only nontrivial contribution comes from the length 2 part. The density matrix $\rho_2$ associated with (5.20) is given by

$$\rho_2 = \frac{1}{24}\Big(|K\rangle + 2|1,1\rangle\Big)\Big(\langle K| + 2\langle 2,2|\Big) + \frac{1}{24}\Big(|K\rangle + 2|2,2\rangle\Big)\Big(\langle K| + 2\langle 1,1|\Big) + \\ + \frac{1}{12}\Big(|1,2\rangle + |2,1\rangle\Big)\Big(\langle 1,2| + \langle 2,1|\Big) \tag{5.21}$$

where $|K\rangle = \sum_{m=1}^{6} |m,m\rangle$. Plugging $\gamma_{\mathcal{O}_{(12)}} = 0$, $\gamma_{K_i} = \frac{\lambda}{2\pi^2}$ and $\mathrm{Tr}_2(H_2\rho_2) = \frac{16}{3}$ into (5.12), $c^1_{\mathcal{O}_{(12)}K_1K_2}$ is found to be

$$\lambda c^1_{\mathcal{O}_{(12)}K_1K_2} = -\frac{1}{2}(\gamma_{K_1} + \gamma_{K_2}) + \frac{\lambda}{16\pi^2}\,\mathrm{Tr}_2(H_2\rho_2) = -\frac{1}{6\pi^2}\lambda\,. \tag{5.22}$$

One can easily check that the conformal constraints (3.17) are satisfied for this example.

In a similar manner, we can compute various $c^1_{123}$ for operators in (5.18). Below we list the non-vanishing combinations of $c^1_{123}$

$$\begin{aligned} c^1_{K\mathcal{O}_{(ij)}\mathcal{O}_{(ij)}} &= -\frac{3}{8\pi^2} & \qquad c^1_{K\mathcal{O}_{(123)}\mathcal{O}_{(123)}} &= -\frac{3}{8\pi^2} \\ c^1_{KK_iK_i} &= -\frac{17}{24\pi^2} & \qquad c^1_{K\mathcal{O}_{[123]}\mathcal{O}_{[123]}} &= -\frac{7}{8\pi^2} \\ c^1_{\mathcal{O}_{(ij)}K_iK_j} &= -\frac{1}{6\pi^2} & \qquad c^1_{\mathcal{O}_{(12)}\mathcal{O}_{[134]}\mathcal{O}_{[234]}} &= -\frac{1}{2\pi^2}\,. \end{aligned} \tag{5.23}$$

We can also check that the following combinations vanish as expected from the non-renormalization theorem

$$c^1_{\mathcal{O}_{(12)}\mathcal{O}_{(23)}\mathcal{O}_{(31)}} = c^1_{\mathcal{O}_{(12)}\mathcal{O}_{(134)}\mathcal{O}_{(234)}} = 0\,. \tag{5.24}$$

Another interesting set of operators is the BMN operators [28]. The exact two impurity BMN operators which diagonalize the planar one-loop dilatation operator were constructed by Beisert [29]

$$\mathcal{B}^{J,n}_{(34)} = \frac{1}{\lambda^{J/2+1}}\sum_{k=0}^{J}\cos\frac{\pi n(2k+1)}{J+1}\,\mathrm{tr}\,\phi^3 Z^k \phi^4 Z^{J-k}\,, \tag{5.25}$$

where $Z = \phi^1 + i\phi^2$. The one-loop anomalous dimension of this operator is $\gamma_{\mathcal{B}^{J,n}_{(34)}} = \frac{\lambda}{\pi^2}\sin^2[\frac{\pi n}{J+1}]$. Vanishing of the anomalous dimension for $n=0$ reflects the fact that $\mathcal{B}^{J,n=0}_{(34)}$ is a 1/2 BPS operator. Using our formula (5.10), we compute some examples of $c^1_{123}$ involving the BMN operators:

$$\begin{aligned} c^1_{\mathcal{O}_{(12)}\mathcal{B}^{J,n}_{(34)}\overline{\mathcal{B}}^{J,n}_{(34)}} &= -\frac{1}{2\pi^2}\sin^2\frac{\pi n}{J+1} \\ c^1_{K\mathcal{B}^{J,n}_{(34)}\overline{\mathcal{B}}^{J,n}_{(34)}} &= -\frac{3}{8\pi^2} - \frac{2}{\pi^2}\frac{1}{J+2}\sin^2\frac{\pi n}{J+1}\left(1 - \frac{1}{J+1}\cos^2\frac{\pi n}{J+1}\right)\,. \end{aligned} \tag{5.26}$$

It is interesting to note that $c^1_{(\mathrm{Konishi})(\mathrm{BPS})(\mathrm{BPS})}$ is independent of the choice of BPS operators,

$$c^1_{K\mathcal{O}_{(12)}\mathcal{O}_{(12)}} = c^1_{K\mathcal{O}_{(123)}\mathcal{O}_{(123)}} = c^1_{K\mathcal{B}^{J,0}_{(34)}\overline{\mathcal{B}}^{J,0}_{(34)}} = -\frac{3}{8\pi^2}\,. \tag{5.27}$$



JHEP08(2004)055

## 6. Relation to closed string field theory

According to the AdS/CFT correspondence, a $n$-point correlation function of single trace operators in $\mathcal{N}=4$ SYM theory corresponds to an on-shell $n$-particle amplitude in the bulk type-IIB string theory on $AdS_5 \times S^5$. The perturbative SYM theory with small 't Hooft coupling $\lambda$ is dual to the bulk string theory in highly curved background with curvature radius $R_{AdS} \sim \lambda^{\frac{1}{4}}\sqrt{\alpha'}$. In this regime of coupling, it is very difficult to compare the SYM results with those of string theory, because the worldsheet theory is a strongly coupled nonlinear sigma model with RR flux. One exception is the sector of SYM carrying large R-charge which corresponds to the string theory on a pp-wave background geometry [28]. The worldsheet theory in pp-wave background becomes free in the light-cone gauge, and it was shown that the spectrum on the bulk side is reproduced by the so-called BMN operators in the SYM side [28]. Moreover, the string interactions described by the light-cone string field theory [30] agree with the three-point functions on the SYM side. In the light cone gauge, the light-cone momentum is conserved $p_1^+ = p_2^+ + p_3^+$. On the SYM side, this corresponds to the conservation of the length of spin chains $L_1 = L_2 + L_3$. Namely, the three-string interactions in the light-cone gauge are dual to the extremal three-point functions.

For general correlators in $\mathcal{N}=4$ SYM theory outside of the BMN sector, there is currently no string theory result to compare. Below, we speculate on some features of string interactions in $AdS_5 \times S^5$ based on our computation of three-point functions in the SYM side. It is natural to expect that the non-extremal correlators in the SYM side correspond to the covariant three string interactions in $AdS_5 \times S^5$. In fact, the pattern of index contraction in free SYM theory in figure 4 fits nicely with the Witten type vertex in closed string field theory as shown in figure 8 [31, 32]. This vertex is a natural generalization of the mid-point interaction vertex in open string field theory. There are two special points $P_{1,2}$ in this three-string vertex which are the counterpart of the mid-point in open string field theory. The points $P_{1,2}$ divide each closed string into two segments with lengths $r, s, t$ given by (4.8) and the closed strings are glued along these segments. One can easily see the correspondence between the "triangular" type diagram of free planar three-point contractions in figure 4 and the Witten type three-string overlap vertex in figure 8. The point $P_1$ and $P_2$ corresponds respectively to the area inside and outside the "triangle" in figure 4.

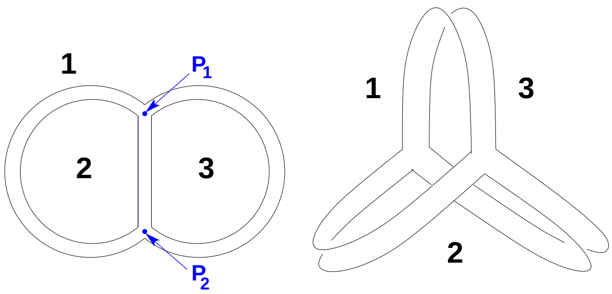


**Figure 8:** Witten type three-string vertex in closed string field theory. The two interaction points $P_{1,2}$ are the analogues of the mid-point in open string field theory.

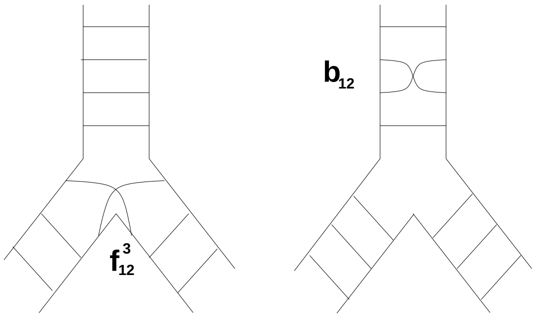


**Figure 9:** Zoom-up of one of the two interaction points $P_{1,2}$. $(a)$ Three-point interactions $f^i_{jk}$ are localized at the interaction points. $(b)$ Two-point interactions $b_{ij}$ are along the overlapping segments.



We found that the structure constant $c_{123}$ gets corrected from the free theory value by the amount $\lambda\, c^1_{123} = \frac{\lambda}{32\pi^2}\left(f^1_{23} + f^2_{31} + f^3_{12}\right)$ when we turned on the 't Hooft coupling $\lambda$. The two terms in the definition of $f^i_{jk}$ in (4.13) corresponds to the two interaction points $P_{1,2}$ and both have the form of the hamiltonian, $\mathcal{K}+2\mathcal{I}-2\mathcal{P}$, acting on one link at the interaction point, as shown in figure 9$a$. This reminds us of the mid-point insertion of operators in open string field theory. In Witten's open string vertex, the only natural place to insert an operator without breaking (naive) associativity is the mid-point. We expect that $P_{1,2}$ are the natural points to insert an operator also for the closed string case, although the product of closed string fields is non-associative. In the bulk string theory side, turning on the 't Hooft coupling $\lambda$ corresponds to changing the radius of $AdS_5 \times S^5$ by adding some operator $\lambda \int d^2z \mathcal{V}$ to the worldsheet action $S$. The form of $c^1_{123}$ suggests that the closed three-string vertex $|V_3^{(\lambda=0)}\rangle$ at $\lambda = 0$ is modified by $\lambda\Big(\widetilde{\mathcal{V}}(P_1) + \widetilde{\mathcal{V}}(P_2)\Big)|V_3^{(\lambda=0)}\rangle$, where $\widetilde{\mathcal{V}}$ is some vertex operator on the worldsheet which can be thought of as the continuum version of $f^i_{jk}$. It would be interesting to know the precise relation between $\mathcal{V}$ and $\widetilde{\mathcal{V}}$. Alternatively, the SYM theory provides us with a complementary description of this deformation. The alternative expression (3.22) implies that instead of inserting an operator at $P_{1,2}$, we can insert open spin chain like interactions in the bulk of closed string worldsheet and treat $P_{1,2}$ as defects. This is shown in figure 9$b$. To our knowledge, this type of deformation of three-string vertex has not been studied. We suspect that supersymmetry and conformal symmetry will again play an important role for understanding this complementarity from the string theory point of view.

## 7. Discussions

In this paper, we have calculated the one-loop correction to the three-point function coefficient in $\mathcal{N} = 4$ SYM theory. In general, this is not identical to the $O(\lambda)$ correction. At the one-loop level, operators mixes with coefficients independent of $\lambda$. In particular, the diagonalization problem of the spin chain hamiltonian (5.4) does not involve $\lambda$. However, at the two-loop level, this is no longer the case. The anomalous dimension matrix is now $O(\lambda^2)$, and correspondingly, the eigen-operators may become $\lambda$ dependent at this order: $\mathcal{O}_\alpha = \mathcal{O}_\alpha^{(0)} + \lambda\mathcal{O}_\alpha^{(1)} + O(\lambda^2)$. This $\lambda$ dependent term in the eigen-operator affects the order $\lambda$ correction to the three-point coefficient through the free contraction such as $\lambda\langle\mathcal{O}_\alpha^{(1)}\mathcal{O}_\beta^{(0)}\mathcal{O}_\rho^{(0)}\rangle_{\text{free}}$. We emphasize that this correction is a two-loop effect which we do not consider in this paper. With regards to planar integrability, this suggests that $O(\lambda)$ integrability requires two-loop integrability.

However, for one-loop planar integrability, the one-loop correction is the natural one to consider as it directly relates to integrable spin chains. The eigenstates of the spin chain hamiltonian (i.e. the first order anomalous dimension matrix) correspond to the eigen-operators we considered in this paper. Interestingly, the calculation of the one-loop correction to the three-point coefficient suggests a new mechanism for closed spin chains — that is, a chain can split into two chains, or conversely, that two chains can "glue" together to form a third chain. Although such processes seem natural from the field theory point of view, to our knowledge, such notions have not been studied within the context of spin chains. The extremal case would correspond to the gluing and splitting process that conserve the total number of spin sites. And the energy cost of such processes would be provided by the one-loop three-point function correction. For the non-extremal case, the total number of spin sites would not be conserved.

As for planar $\mathcal{N} = 4$ SYM theory, we have shown that one-loop correction for both anomalous dimension and three-point coefficient can be calculated using spin chains. We have however not exploited integrable techniques for open spin chain that may possibly be useful for calculating non-extremal three-point coefficients of large scaling dimension operators. Because of the presence of the density matrices, it is not clear to us what is the most efficient manner to utilize integrability.

The usefulness of the integrable SO(6) spin chain for planar $\mathcal{N} = 4$ SYM derives from the fact that the values of interest thus far are all solely dependent on the SO(6) indices of the scalar fields. It would be interesting to see whether the chains of indices (or the spin chain data) by themselves are sufficient to directly determine the planar four- or higher-point functions whose spatial dependence is not completely constrained by conformal symmetry. From the point of view of AdS/CFT correspondence, $n$-point functions in the SYM side correspond to $n$-string interactions in $AdS_5 \times S^5$. In the covariant closed string field theory of non-polynomial type [33, 34], it is known that $n$-string interaction vertices with arbitrarily high $n$ are required in order to cover the whole moduli space of closed Riemann surface with punctures. It would be interesting to relate this with the structure of $n$-point functions in SYM theory.

There are additionally several directions to extend our work. One natural extension is to calculate three-point functions of more general operators outside the SO(6) subsector. The original result of Minahan and Zarembo for the planar one-loop anomalous dimension in the SO(6) subsector has been extended to the general PSU(2, 2|4) sector in [3, 4, 5]. It would be interesting to repeat our analysis of the three-point functions for the PSU(2, 2|4) sector and see if a similar role is played by an analogous PSU(2, 2|4) open spin chain. It would also be interesting to study the higher loop corrections to the three-point function coefficients and their relations to integrable spin chains.



JHEP08(2004)055

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## A. Computation of $B$ and $F$

The function $B(x_1, x_2)$ as defined in (3.3) is given by

$$\begin{aligned} B(x_1,x_2) &= 2\lambda \int d^4y \, \frac{[8\pi^2|x_{12}|^2]^2}{[8\pi^2|y-x_1|^2]^2\,[8\pi^2|y-x_2|^2]^2} \\ &= \frac{2\lambda}{(8\pi^2)^2}|x_{12}|^4 \int d^4y \, \frac{1}{|y-x_1|^4|y-x_2|^4} \,. \end{aligned} \tag{A.1}$$

We regularize the propagator by differential regularization [35] where a small distance cutoff is inserted into the propagator. Explicitly,

$$\Delta(y-x) = \frac{1}{(y-x)^2+\epsilon^2} \,. \tag{A.2}$$

The integral in (A.1) is easily evaluated as

$$\int d^4y \, \Delta(y)^2 \Delta(y-x)^2 = \pi^2 \int_0^1 dt \, \frac{t(1-t)}{[t(1-t)x^2+\epsilon^2]^2} = \frac{2\pi^2}{x^4}\Big[\log\frac{x^2}{\epsilon^2} - 1\Big] \,. \tag{A.3}$$

Letting $\Lambda^2 = 1/\epsilon^2$, this gives

$$B(x_1,x_2) = \frac{\lambda}{16\pi^2}\left[\ln|x_{12}\Lambda|^2 - 1\right] . \tag{A.4}$$

$F(x_3; x_1, x_2)$ is defined in (3.5) and takes the form

$$\begin{aligned} F(x_3;x_1,x_2) &= 2\lambda \int d^4y \, \frac{[8\pi^2|x_{13}|^2][8\pi^2|x_{23}|^2]}{[8\pi^2|y-x_3|^2]^2\,[8\pi^2|y-x_1|^2][8\pi^2|y-x_2|^2]} \\ &= \frac{2\lambda}{(8\pi^2)^2}|x_{13}|^2|x_{23}|^2 \int d^4y \, \frac{1}{|y-x_3|^4|y-x_1|^2|y-x_2|^2} \,. \end{aligned} \tag{A.5}$$

Using differential regularization, we need to calculate the integral $I_3$ defined by

$$I_3 = \int d^4y \, \Delta(y)^2 \Delta(y-x_1)\Delta(y-x_2) \,. \tag{A.6}$$

To compute $I_3$, it is useful to use the inversion

$$x^\mu \to \widehat{x}^\mu = \frac{x^\mu}{x^2} \,. \tag{A.7}$$

Obviously, this map is involutive $\widehat{\widehat{x^\mu}} = x^\mu$, and the norm is transformed as

$$(\widehat{x})^2 = \frac{1}{x^2} \,, \qquad (\widehat{x}_1 - \widehat{x}_2)^2 = \frac{(x_1-x_2)^2}{x_1^2 x_2^2} \,. \tag{A.8}$$

The measure and the propagator transform as

$$d^4\widehat{z} = \frac{d^4z}{(z^2)^4}\,, \qquad \Delta(\widehat{z}-x) = \Delta(\widehat{z}-\widehat{\widehat{x}}) = \frac{1}{x^2}\frac{z^2}{(z-\widehat{x})^2+\epsilon^2 z^2\widehat{x}^2}\,. \tag{A.9}$$

By changing the integration variable $y \to \widehat{z}$, the integral $I_3$ becomes

$$\begin{aligned} I_3 &= \int d^4\widehat{z}\,\Delta(\widehat{z})^2\Delta(\widehat{z}-x_1)\Delta(\widehat{z}-x_2) \\ &= \frac{1}{x_1^2x_2^2}\int d^4z\,\frac{1}{(1+\epsilon^2z^2)^2}\frac{1}{(z-\widehat{x}_1)^2+\epsilon^2z^2\widehat{x}_1^2}\frac{1}{(z-\widehat{x}_2)^2+\epsilon^2z^2\widehat{x}_2^2}\,. \end{aligned} \tag{A.10}$$

Observing that the logarithmic divergence is coming from $z=\infty$, up to $\mathcal{O}(\epsilon)$ we can replace $I_3$ by

$$\begin{aligned} I_3 &= \frac{1}{x_1^2x_2^2}\int d^4z\,\frac{1}{(1+\epsilon^2z^2)^2}\frac{1}{z^2}\frac{1}{(z-(\widehat{x}_1-\widehat{x}_2))^2} \\ &= \frac{1}{x_1^2x_2^2}\int d^4z\,\frac{1}{(1+z^2)^2}\frac{1}{z^2}\frac{1}{(z-\epsilon(\widehat{x}_1-\widehat{x}_2))^2}\,. \end{aligned} \tag{A.11}$$

Using Feynman parameter, $I_3$ is written as

$$\begin{aligned} I_3 &= \frac{\pi^2}{x_1^2x_2^2}\int_0^1 dt\int_0^{1-t} ds\,s\left[s+t(1-t)\epsilon^2(\widehat{x}_1-\widehat{x}_2)^2\right]^{-2} \\ &= \frac{\pi^2}{x_1^2x_2^2}\int_0^1 dt\left[\log\left(\frac{1+t\epsilon^2(\widehat{x}_1-\widehat{x}_2)^2}{t\epsilon^2(\widehat{x}_1-\widehat{x}_2)^2}\right)-\frac{1}{1+t\epsilon^2(\widehat{x}_1-\widehat{x}_2)^2}\right] \\ &= -\frac{\pi^2}{x_1^2x_2^2}\log\epsilon^2(\widehat{x}_1-\widehat{x}_2)^2+\mathcal{O}(\epsilon) \\ &= \frac{\pi^2}{x_1^2x_2^2}\log\frac{x_1^2x_2^2}{\epsilon^2(x_1-x_2)^2}\,. \end{aligned} \tag{A.12}$$

Applying this result into (A.5) and again taking $\Lambda^2 = 1/\epsilon^2$, we obtain

$$F(x_3;x_1,x_2) = \frac{\lambda}{32\pi^2}\ln\left|\frac{x_{13}x_{23}\Lambda}{x_{12}}\right|^2\,. \tag{A.13}$$

Introducing the renormalization scale $\mu$, we can renormalize $B$ and $F$ for example by taking $\mu=\Lambda$,

$$B(x_1,x_2) = b_0+\frac{\lambda}{16\pi^2}\ln|x_{12}\mu|^2\,, \qquad F(x_3;x_1,x_2) = f_0+\frac{\lambda}{32\pi^2}\ln\left|\frac{x_{13}x_{23}\mu}{x_{12}}\right|^2\,. \tag{A.14}$$

where $b_0 = -\frac{\lambda}{16\pi^2}$ and $f_0=0$ for differential regularization but it is useful here to leave it arbitrary. Under the rescaling $\mu\to\mu e^a$, $b_0$ and $f_0$ are shifted as follows,

$$b_0' = b_0+\frac{\lambda a}{8\pi^2}\,, \qquad f_0' = f_0+\frac{\lambda a}{16\pi^2}\,. \tag{A.15}$$

Clearly, the combination $2f_0'-b_0' = 2f_0-b_0$ is independent of the renormalization scale of $\mu$. Thus, from the explicit calculation,

$$2f_0-b_0 = \frac{\lambda}{16\pi^2} \tag{A.16}$$

is scheme independent.

## B. Index contraction as a matrix integral

The free contraction coefficient $\tilde{C}^0$ appeared in the text can be written as a large-$N$ planar matrix integral

$$\Big\langle\Big\langle \mathrm{Tr}(X^{i_1}\cdots X^{i_{L_1}})\,\mathrm{Tr}(Y^{j_1}\cdots Y^{j_{L_2}})\,\mathrm{Tr}(Z^{k_1}\cdots Z^{k_{L_3}})\Big\rangle\Big\rangle_{\text{planar}} \tag{B.1}$$

where the matrix model expectation value $\langle\langle\mathcal{O}\rangle\rangle$ is defined by

$$\begin{aligned} \langle\langle\mathcal{O}\rangle\rangle &= \int \prod_{m=1}^{6} dX^m dY^m dZ^m\, \mathcal{O} e^{-S_{3\text{pt}}}\,, \\ S_{3\text{pt}} &= \frac{1}{4}\sum_{m=1}^{6} \mathrm{Tr}\left[(X^m)^2+(Y^m)^2+(Z^m)^2-2X^mY^m-2Y^mZ^m-2Z^mX^m\right]. \end{aligned} \tag{B.2}$$

$X^m, Y^m$ and $Z^m$ are $N\times N$ hermitian matrices. The propagator of these matrices derived from the action (B.2) is

$$\langle\langle (X^m)^a_b (Y^n)^c_d\rangle\rangle = \langle\langle (Y^m)^a_b (Z^n)^c_d\rangle\rangle = \langle\langle (Z^m)^a_b (X^n)^c_d\rangle\rangle = \delta^{m,n}\delta^a_d\delta^c_b\,, \tag{B.3}$$

where $a,b,c,d$ are the color indices. These propagators connect the matrices in (B.1) without the self-contraction inside the trace. In the planar limit, the resulting contraction is exactly the same as that appears in the computation in the free gauge theory.

The one-loop corrected three-point function coefficient would be obtained by adding interactions to the matrix model. Note that the one loop corrected dilatation operator obtained in [3] can be written as a matrix model

$$\begin{aligned} \Gamma^{i_1\cdots i_L}_{j_1\cdots j_L} &= \int \prod_{m=1}^{6} dX^m dY^m\ \mathrm{Tr}(X^{i_1}\cdots X^{i_L})\,\mathrm{Tr}(Y^{j_1}\cdots Y^{j_L}) e^{-S_{2\text{pt}}}\,, \\ S_{2\text{pt}} &= \mathrm{Tr}(X^mY^m)+\frac{g^2}{16\pi^2}\mathrm{Tr}[X^m,X^n][Y^m,Y^n]+\frac{g^2}{32\pi^2}\mathrm{Tr}[X^m,Y^n][X^m,Y^n]. \end{aligned} \tag{B.4}$$

Similarly, the three-point function coefficients are reproduced by the matrix model interaction

$$\begin{aligned} S^{(int)}_{3\text{pt}} &= 2\,\mathrm{Tr}[X^m,X^n][Y^m,Z^n]+\mathrm{Tr}[X^m,Y^n][X^m,Z^n]+ \\ &\quad +2\,\mathrm{Tr}[Y^m,Y^n][Z^m,X^n]+\mathrm{Tr}[Y^m,Z^n][Y^m,X^n]+ \\ &\quad +2\,\mathrm{Tr}[Z^m,Z^n][X^m,Y^n]+\mathrm{Tr}[Z^m,X^n][Z^m,Y^n]. \end{aligned} \tag{B.5}$$

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JHEP08(2004)055

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